Finite Frame Theory

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Suggested prerequisites. *Linear algebra: bases, eigenvalues, eigenvectors, inner products. Complex variables: basic properties of complex numbers.*

1 Introduction

Given a signal, whether it is a discrete vector or a continuous function, one desires to write it in terms of simpler components. Typically, these components or "building blocks" form what is called a *basis*. A basis is an optimal set, having the minimal number of elements, such that any vector (signal) in the underlying space can be written *uniquely* as a linear combination of the basis vectors. Bases play an important role in the analysis of vector spaces, since the characteristics of the signal can be read off from the coefficients in the basis representation. However, if the coefficients get corrupted by noise or if some get lost during transmission then valuable signal information can get lost beyond recovery. The main problem with bases is this lack of flexibility - even a slight modification of a basis can leave us with a set that is no longer a basis. Since the basis representation is typically nonredundant one can try to bring in more flexibility by adding some extra elements and sacrificing the uniqueness property of a basis representation. This leads to the notion of a *frame*. A frame can be thought of as a *redundant* basis, having more elements than needed. In fact, in any finite dimensional vector space every finite spanning set is a frame. Although the redundancy of a frame leads to non-unique representations, this also makes the corresponding signal representations resilient to noise and robust to transmission losses. In applications, such robustness might be more desirable than having a unique representation.

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It is widely acknowledged that the idea of frames originated in the 1952 paper by Duffin and Schaeffer [11], but frames have only gained significant popularity relatively recently due to work such as [10]. Frames are now standard tools in signal processing and are of great interest to mathematicians and engineers alike. This chapter focuses on frames in finite dimensional spaces. The notion of frames for infinite dimensional spaces like function spaces is far more subtle [6, 9, 26] and will not be discussed here. Along with introducing finite frame theory, this chapter discusses a special highly desirable class of frames called *equiangular tight frames.* Possible research ideas suitable for an undergraduate curriculum are also discussed.

Throughout the chapter, many of the well known results will be stated without proofs and the reader will be provided with the necessary references. It is assumed that readers have taken some undergraduate linear algebra course, and are familiar with the notion of a basis and its fundamental properties. For an in depth study of linear algebra, readers may refer to [12, 17]. A nominal knowledge of complex numbers is also assumed. In all that follows, $\mathbb{R}$ will denote the set of real numbers, and $\mathbb{C}$ will denote the set of complex numbers. For a given $c \in \mathbb{C}$, the complex conjugate of c is denoted by $\overline{c}$, and the modulus of c is denoted by $|c|$. In a setting that applies to both $\mathbb{R}$ and $\mathbb{C}$ we will use the notation $\mathbb{F}$, the elements of which are called *scalars*.

We first recall one of the most significant properties of a basis in the following result [12].

Theorem 1. *[12] Let $\mathscr{V}$ be a vector space and $B = \{v_1, v_2, \ldots, v_n\}$ be a subset of $\mathscr{V}$. Then B is a basis for $\mathscr{V}$ if and only if each $v \in \mathscr{V}$ can be uniquely expressed as a linear combination of vectors of B, that is, there exist unique scalars $c_1, c_2, \ldots, c_n$ such that*

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n. \tag{1}$$

The scalars c_i in (1) are called the *coefficients* of v with respect to B. Note that each basis for a given vector space has the same number of elements, and this number is the dimension of the underlying vector space. As will become apparent later, in the case of a frame there is an added flexibility in that frames for the same space can differ in the number of elements. The computation of the coefficients in (1) is important since this allows us to represent v in terms of the basis elements. However, this process can be cumbersome. The concept of an inner product can greatly simplify these calculations. For the convenience of the reader, we next recall some standard definitions and properties pertinent to inner products.

Definition 1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. An inner product on $\mathscr{V}$ is a function that assigns to every ordered pair of vectors $u, v \in \mathscr{V}$, a scalar in $\mathbb{F}$ denoted by $\langle u, v\rangle$, such that for all $u, v, w \in \mathscr{V}$ and all $c \in \mathbb{F}$, the following hold:

(a) $\langle u+w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$.
(b) $\langle cu, v\rangle = c\langle u, v\rangle$.
(c) $\overline{\langle u, v\rangle} = \langle v, u\rangle$, where the bar denotes complex conjugation.
(d) $\langle u, u\rangle \geq 0$, with equality if and only if $u = 0$.

A vector space endowed with an inner product is called an *inner product space*. If $\mathbb{F} = \mathbb{R}$, it is called a *real inner product space* and if $\mathbb{F} = \mathbb{C}$, it is called a *complex inner product space*.

Example 1. In $\mathbb{F}^n$, an inner product of two vectors $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ can be defined as

$$\langle u, v \rangle = \sum_{i=1}^{n} a_i \overline{b}_i.$$

This is the standard inner product of $\mathbb{F}^n$. When $\mathbb{F} = \mathbb{R}$, the conjugations are not needed, and we have what is commonly referred to as the *dot product*, often written as $u \cdot v$.

The definition of an inner product is used to generalize the notion of length in a vector space. Recall that in $\mathbb{R}^3$, the Euclidean length of a vector $v = (a, b, c)$ is given by $\sqrt{a^2 + b^2 + c^2} = \sqrt{\langle v, v \rangle}$. This leads to the following.

Definition 2. Let $\mathscr{V}$ be an inner product space. For $v \in \mathscr{V}$, the *norm* or *length* of v is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

If $\|v\| = 1$ then v is called *unit normed*.

Definition 3. Two distinct vectors u, v in an inner product space $\mathscr{V}$ are said to be *orthogonal* if $\langle u, v \rangle = 0$. A subset S of $\mathscr{V}$ is called an *orthogonal set* if any two distinct vectors in S are orthogonal.

Recall that in $\mathbb{R}^2$ or $\mathbb{R}^3$ two vectors that are mutually perpendicular to each other have dot product equal to zero. The notion of an inner product can thus be used to infer the (angular) distance between vectors in a vector space, and leads to a special kind of basis called an *orthonormal basis*.

Definition 4. A basis B of a vector space $\mathscr{V}$ is called an *orthonormal basis* (ONB) if B is an orthogonal set in which every vector is unit normed.

Every finite dimensional vector space has an orthonormal basis. This is a consequence of the Gram-Schmidt orthogonalization process [12]. One of the main advantages of an ONB is that the coefficients in the basis representation, when using an ONB, are very easy to compute. This is due to the following result.

Theorem 2. *[12] Let $B = \{u_i\}_{i=1}^n$ be an ONB of $\mathscr{V}$. Any vector $v \in \mathscr{V}$ can be written in terms of the vectors in B as*

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i.$$

Theorem 2 shows that the unique coefficients in the basis representation in (1), when using an ONB, are just given by the inner products $\langle v, u_i \rangle$, and therefore very simple to calculate. Theorem 2 leads to Parseval's Formula which can be thought of as a generalization of the Pythagorean Identity.

Proposition 1 (Parseval's Formula). *[15] Let $\{u_i\}_{i=1}^n$ be an ONB of a vector space $\mathscr{V}$. Then for every $v \in \mathscr{V}$*

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, u_i \rangle|^2. \tag{2}$$

Parseval's Formula is particularly intimate to frame theory, and the importance of this can be understood by taking a closer look at (2). It says that the norm of the signal v is completely determined by the orthonormal basis coefficients $\{\langle v, u_i \rangle\}$. Suppose that the signal v cannot be analyzed directly but one can measure the coefficients $\{\langle v, u_i \rangle\}$. Since both sides of (2) have the meaning of energy, this suggests that some valuable information of the signal can be obtained solely from its coefficients even if one does not know what the signal is.

A natural question is: why do we wish to generalize bases or why do we want to look beyond bases? First of all, it might be worth pointing out that once a basis for a vector space $\mathscr{V}$ has been fixed, for each $v \in \mathscr{V}$, one can just work with the coefficients of v that appear in the basis representation. This means that if $B = \{v_1, v_2, \dots, v_n\}$ is a basis of $\mathscr{V}$ and if $c_1, c_2, \dots, c_n$ are the unique coefficients representing v, that is,

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, \tag{3}$$

then in order to store or transmit v, one only uses the vector $(c_1, c_2, \dots, c_n)$. Once these coefficients are known, one can recover v using (3).

Now suppose that in transmitting v, the coefficients $\{c_i\}_{i=1}^n$ get corrupted by noise, and as a result what is received is $\{c_i + \mu_i\}_{i=1}^n$. During the recovery process, one obtains

$$\widehat{v} = \sum_{i=1}^{n} (c_i + \mu_i) v_i = \sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{n} \mu_i v_i = v + \varepsilon.$$

Instead of a basis suppose that one uses a spanning set, that is, a set that spans $\mathscr{V}$ but is linearly dependent. Such a set could be $\{v_1, \dots, v_n, v_{n+1}, \dots, v_m\} = \{v_i\}_{i=1}^m$, $m > n$, obtained from B by adding some additional vectors of $\mathscr{V}$. Then v can be recovered by calculating

$$\widehat{v} = \sum_{i=1}^{m} c_i v_i + \sum_{i=1}^{m} \mu_i v_i$$

where $c_i = 0$, $n < i \leq m$. Since $\{v_i\}_{i=1}^m$ is a linearly dependent set, there is a possibility that the second summation $\sum_{i=1}^m \mu_i v_i$ will become zero, thereby canceling the noise, something that is never possible when using a basis. Intuitively, this shows how the effect of noise can be reduced by using a redundant set.

As another instance of the benefit of having a redundant spanning set, let us consider $v = (1,1) \in \mathbb{R}^2$. [1] Using the standard orthonormal basis of $\mathbb{R}^2$, $\{e_1 = (1,0), e_2 = (0,1)\}$, v can be written as

$$v = 1 \cdot e_1 + 1 \cdot e_2.$$

[1] The authors would like to thank Andy Kebo for sharing this example.

If one of the coefficients, say the second coefficient is lost, then at the reconstruction stage one gets

$$\widehat{v} = 1 \cdot e_1 + 0 \cdot e_2 = (1,0)$$

and the error in the reconstruction is

$$\|v - \widehat{v}\|_2 = 1$$

where $\|.\|_2$ is the Euclidean distance in $\mathbb{R}^2$. Instead of an ONB, let us now use the redundant set

$$\{f_k = (\cos(2k\pi/6), \sin(2k\pi/6))\}_{k=0}^{5}.$$

See Fig. 1(a). In terms of this set, the vector $(1,1)$ can be written as

$$v = 0.333f_0 + 0.455f_1 + 0.122f_2 - 0.333f_3 - 0.455f_4 - 0.122f_5.$$

If the coefficient corresponding to f_0 is lost then one obtains

$$\widehat{v} = 0.455f_1 + 0.122f_2 - 0.333f_3 - 0.455f_4 - 0.122f_5$$

and the reconstruction error is

$$\|v - \widehat{v}\|_2 = 1/3$$

which is less than the error when the first coefficient was lost while using the standard ONB of $\mathbb{R}^2$. Losing a single coefficient with an ONB in $\mathbb{R}^2$ can be thought of as losing 50% of the coefficients. In the case of the redundant set under consideration suppose that 50%, that is, three coefficients are lost. If they are the first three, then the reconstructed vector is

$$\widehat{\widehat{v}} = -0.333f_3 - 0.455f_4 - 0.122f_5$$

and the reconstruction error is

$$\|v - \widehat{\widehat{v}}\|_2 = 1/\sqrt{2}$$

which is still less than when using the standard ONB of $\mathbb{R}^2$.

In the above, what we have sacrificed by using a linearly dependent spanning set instead of a basis is that we no longer have the unique representation of Theorem 1. The question is whether a unique representation is necessary for our purpose and the answer is no; as long as we are able to represent every vector in terms of the set, it does not matter. This notion of adding redundancy is what is incorporated in a *frame* for a finite dimensional vector space. The set $\{f_k = (\cos(2k\pi/6), \sin(2k\pi/6))\}_{k=0}^{5}$ used in the discussion above is not a basis but is a frame for $\mathbb{R}^2$, and we have shown how redundancy is better when we have transmission losses. The next section gives the basics of finite frame theory. The reader is urged to read [6, 15] for more details. An excellent overview of the ideas underlying frames can be found in [18].

2 Frames in Finite Dimensional Spaces

To help us think of frames as generalizations of bases, let us look at Parseval's Formula in Proposition 1. Relaxing the condition in (2) gives the following definition.

Definition 5. Let $\mathscr{V}$ be an inner product space and let $\{f_i\}_{i\in\mathscr{I}}$ be a subset of $\mathscr{V}$ indexed by some countable set $\mathscr{I}$.

(i) The set $\{f_i\}_{i\in\mathscr{I}}$ is a *frame* if there exist constants $0 < A \le B < \infty$ such that for every $v \in \mathscr{V}$,

$$A\|v\|^2 \le \sum_{i\in\mathscr{I}} |\langle v, f_i\rangle|^2 \le B\|v\|^2. \tag{4}$$

(ii) The constants A and B are called the *lower* and *upper frame bound*, respectively.
(iii) If $A = B$, the frame is called a *tight frame*.
(iv) If for each $i \in \mathscr{I}$, $\|f_i\| = 1$, the frame is called a *unit normed frame*.

Due to Parseval's Formula, an orthonormal basis is a unit normed tight frame with frame bound equal to 1. In a finite d-dimensional vector space $\mathscr{V}$, a finite set $\{f_i\}_{i=1}^n$, $n \ge d$, is a frame if and only if $\{f_i\}_{i=1}^n$ is a spanning set of $\mathscr{V}$ [6].

Let $\{f_i\}_{i=1}^n$ be a frame for a finite dimensional inner product space $\mathscr{V}$. The *Bessel map* $F : \mathscr{V} \to \mathbb{F}^n$ is defined by

$$F(v) = \{\langle v, f_i\rangle\}_{i=1}^n, \quad v \in \mathscr{V}. \tag{5}$$

The adjoint of F is given by

$$F^* : \mathbb{F}^n \to \mathscr{V}, \quad F^*(\{c_i\}_{i=1}^n) = \sum_{i=1}^n c_i f_i. \tag{6}$$

The mapping F is often referred to as the *analysis operator*, while F^* is referred to as the *synthesis operator*. In a finite dimensional space like $\mathbb{R}^d$ or $\mathbb{C}^d$, the synthesis operator F^* can be written as a $d \times n$ matrix whose columns are the frame vectors. The analysis operator F is then an $n \times d$ matrix whose i^{th} row is f_i^*. In other words, F^* is just the conjugate transpose of F in this setting.

Lemma 1. *The analysis operator F given by (5) is one to one.*

Proof. To show that F is one to one, it is enough to show that the null space of F consists of only the zero vector. Let $Fv = 0$ for some $v \in \mathscr{V}$. Then for $i = 1, \dots, n$, $\langle v, f_i\rangle = 0$. Since $\{f_i\}_{i=1}^n$ spans $\mathscr{V}$, v must equal zero. Thus F is one to one. □

Lemma 2. *If $\{e_i\}_{i=1}^n$ is the standard orthonormal basis [2] of $\mathbb{F}^n$, then for $i = 1, \dots, n$,*

$$F^*(e_i) = f_i.$$

[2] e_i is the vector whose i^{th} coordinate is equal to 1 and the rest are zero.

Proof. Let v be a vector in $\mathscr{V}$. Then for $i = 1, \ldots, n$,

$$\begin{aligned}\langle v, F^* e_i \rangle &= \langle F v, e_i \rangle \\ &= \langle v, f_i \rangle.\end{aligned}$$

Thus $F^* e_i = f_i$. □

Lemma 3. *The synthesis operator given by (6) maps $\mathbb{F}^n$ onto $\mathscr{V}$.*

Proof. Let v be a vector in $\mathscr{V}$. Since $\{f_i\}_{i=1}^n$ spans $\mathscr{V}$, there exist constants $\alpha_1, \ldots, \alpha_n$ such that

$$\begin{aligned}v &= \alpha_1 f_1 + \ldots + \alpha_n f_n \\ &= \alpha_1 F^*(e_1) + \ldots + \alpha_n F^*(e_n) \\ &= F^*(\alpha_1 e_1 + \ldots + \alpha_n e_n).\end{aligned}$$

where the penultimate step follows from Lemma 2. Thus the vector $\alpha_1 e_1 + \ldots + \alpha_n e_n \in \mathbb{F}^n$ gets mapped to v showing that F^* is onto. □

The composition of F^* with F gives the *frame operator*

$$S : \mathscr{V} \to \mathscr{V}, \quad S(v) = F^* F(v) = \sum_{i=1}^{n} \langle v, f_i \rangle f_i. \tag{7}$$

If we restrict ourselves to either $\mathbb{R}^d$ or $\mathbb{C}^d$, then the frame operator S is the $d \times d$ matrix $F^* F$ where F is the matrix corresponding to the analysis operator. Note that in terms of the frame operator, for any $v \in \mathscr{V}$,

$$\sum_{i=1}^{n} |\langle v, f_i \rangle|^2 = \langle S v, v \rangle,$$

and (4) can be rewritten as

$$A\|v\|^2 \leq \langle S v, v \rangle \leq B\|v\|^2. \tag{8}$$

For a tight frame, when $A = B$, (8) implies that

$$S = AI,$$

and so

$$\sum_{i=1}^{n} |\langle v, f_i \rangle|^2 = A\|v\|^2$$

$$\text{or,} \quad \|v\|^2 = \sum_{i=1}^{n} \frac{1}{A} |\langle v, f_i \rangle|^2. \tag{9}$$

Note that (9) resembles Parseval's formula that is satisfied by an ONB. In this regard, unit normed tight frames are the redundant counterparts of orthonormal bases. Finally, since $S = AI$ for a tight frame, (7) implies a representation of any $v \in \mathscr{V}$ in terms of the vectors of a tight frame:

$$v = \frac{1}{A}\sum_{i=1}^{n}\langle v, f_i\rangle f_i. \tag{10}$$

If we define a new set of vectors as

$$g_i = \frac{1}{A} f_i,$$

then (10) can be written as

$$v = \sum_{i=1}^{n}\langle v, g_i\rangle f_i = \sum_{i=1}^{n}\langle v, f_i\rangle g_i. \tag{11}$$

As can be seen from (11), for a tight frame, the coefficients in the expression for v can be obtained simply by calculating the inner products with the g_is or the f_is, in a manner similar to the case of an ONB. In the case of a general frame that is not necessarily tight, a frame representation as in (10) or (11) is still possible and is now given.

Theorem 3. *[6] Let $\{f_i\}_{i=1}^n$ be a frame for a finite dimensional inner product space $\mathscr{V}$, and let S be the frame operator. Then*

(i) S is invertible and self-adjoint, i.e., $S = S^*$.
(ii) Every $v \in \mathscr{V}$ can be represented as

$$v = \sum_{i=1}^{n}\langle v, S^{-1}f_i\rangle f_i = \sum_{i=1}^{n}\langle v, f_i\rangle S^{-1}f_i. \tag{12}$$

Proof. (i) We will first show that the frame operator S is one to one. By (8), $Sv = 0$ forces $v = 0$. Therefore the null space of S contains only the zero vector and S is one-to-one. Since S maps $\mathscr{V}$ to $\mathscr{V}$, S is also onto, and hence invertible.
By properties of the adjoint operator:

$$S^* = (F^*F)^* = F^*(F^*)^* = F^*F = S.$$

Thus S is self-adjoint.

(ii) By (i), S^{-1} exists, and so for each $v \in \mathscr{V}$ we have

$$\begin{aligned} v = S^{-1}Sv &= S^{-1}\sum_{i=1}^{n}\langle v, f_i\rangle f_i, \quad \text{by (7)}, \\ &= \sum_{i=1}^{n}\langle v, f_i\rangle S^{-1}f_i. \end{aligned}$$

Note that since S is self-adjoint, so is S^{-1}. Thus, again by (7),

$$v = SS^{-1}v = \sum_{i=1}^{n} \langle S^{-1}v, f_i \rangle f_i$$
$$= \sum_{i=1}^{n} \langle v, S^{-1}f_i \rangle f_i.$$

□

The set $\{S^{-1}f_i\}_{i=1}^n$ is also a frame for $\mathscr{V}$ and is called the *canonical dual* of $\{f_i\}_{i=1}^n$. The numbers $\langle v, S^{-1}f_i \rangle$ are called the *frame coefficients*.

If $\{f_i\}_{i=1}^n$ is a spanning set that is not a basis of $\mathscr{V}$, there must exist constants $\{\gamma_i\}_{i=1}^n$, not all zero, such that $\sum_{i=1}^n \gamma_i f_i = 0$. Thus, by adding zero to the first part of (12),

$$v = \sum_{i=1}^{n} \langle v, S^{-1}f_i \rangle f_i + \sum_{i=1}^{n} \gamma_i f_i$$
$$= \sum_{i=1}^{n} (\langle v, S^{-1}f_i \rangle + \gamma_i) f_i,$$

and so there are infinitely many representations of v in terms of the frame vectors. The representation of v in (12) is called the *canonical form*. With a frame, we therefore have much more flexibility when choosing a representation for v compared to a basis representation. However, the special feature of the frame coefficients $\{\langle v, S^{-1}f_i \rangle\}$ is that they have the minimal ℓ^2-norm among all coefficients $\{c_i\}_{i=1}^n$ such that $v = \sum_{i=1}^n c_i f_i$. This can be stated as follows.

Theorem 4. *[6] Let $\{f_i\}_{i=1}^n$ be a frame for a finite dimensional inner product space $\mathscr{V}$ with frame operator S. If $v \in \mathscr{V}$ has the representation $v = \sum_{i=1}^n c_i f_i$, then*

$$\sum_{i=1}^{n} |c_i|^2 = \sum_{i=1}^{n} |\langle v, S^{-1}f_i \rangle|^2 + \sum_{i=1}^{n} |c_i - \langle v, S^{-1}f_i \rangle|^2.$$

Example 2 (Computing a frame expansion). Consider the frame in $\mathbb{R}^2$ given by the rows of

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and let

$$v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The frame operator corresponding to this frame is given by

$$S = F^T F = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

and the frame coefficients for v are

$$\langle v, S^{-1} f_1 \rangle = \frac{4}{3}$$
$$\langle v, S^{-1} f_2 \rangle = -\frac{5}{3}$$
$$\langle v, S^{-1} f_3 \rangle = -\frac{1}{3}.$$

One can then verify that

$$v = \frac{4}{3} f_1 - \frac{5}{3} f_2 - \frac{1}{3} f_3,$$

which is the canonical form of v. The ℓ^2-norm of the coefficients in the canonical form is $\sqrt{\frac{14}{3}}$. It is not too difficult to see that v can also be written as $v = f_1 - 2 f_2$. The ℓ^2-norm of the coefficients in this expansion is given by $\sqrt{5}$. As expected, this is greater than the ℓ^2-norm of the coefficients in the canonical form.

Note that computing the frame coefficients involves inverting the frame operator S which can be numerically unstable. In the case of a tight frame, this step is drastically simplified due to the fact that $S^{-1} = \frac{1}{A} I$, a constant multiple of the identity. This feature has made tight frames highly desirable. They emulate ONBs and at the same time provide the benefits of redundancy that come from using frames.

The *frame potential* [2] of the frame $\{f_i\}_{i=1}^n$ is the number $FP(\{f_i\}_{i=1}^n)$ defined by

$$FP(\{f_i\}_{i=1}^n) = \sum_{i=1}^n \sum_{j=1}^n |\langle f_i, f_j \rangle|^2.$$

The frame potential is used to give an important characterization of unit normed tight frames and ONBs.

Theorem 5 (Theorem 6.2, [2]). *Let $d, n \in \mathbb{N}$ with $d \leq n$, and let $\{f_i\}_{i=1}^n$ be a unit normed frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. Then $FP(\{f_i\}_{i=1}^n) \geq \frac{n^2}{d}$ with equality if and only if $\{f_i\}_{i=1}^n$ is a unit normed tight frame (or an ONB in the case $n = d$).*

The optimal lower bound of a frame is the supremum over all constants A that satisfy the left inequality in (4). Similarly, the optimal upper bound is the infimum over all constants B that satisfy the right side of inequality (4). In the finite dimensional setting, the optimal lower and upper frame bounds are given by the minimum and maximum eigenvalues of the matrix of S. The following theorem gives some useful results involving the eigenvalues of the frame operator. The proof, given in [6], is included here.

Theorem 6. *[6] Let $\{f_i\}_{i=1}^n$ be a frame for a d-dimensional space $\mathscr{V}$. Then the following hold.*

(i) The optimal lower frame bound is the smallest eigenvalue of S, and the optimal upper frame bound is its largest eigenvalue.

(ii) Let $\{\lambda_i\}_{i=1}^d$ denote the eigenvalues of S. Then

$$\sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{n} \|f_i\|^2.$$

(iii) If $\{f_i\}_{i=1}^n$ is a tight frame and for all i, $\|f_i\| = 1$, then the frame bound is $A = \frac{n}{d}$.

Proof. (i) Since the frame operator $S : \mathscr{V} \to \mathscr{V}$ is self-adjoint, there is an orthonormal basis of $\mathscr{V}$ consisting of eigenvectors of S. Denote this eigenvector basis by $\{e_i\}_{i=1}^d$ and the corresponding eigenvalues by $\{\lambda_i\}_{i=1}^d$. Every $v \in \mathscr{V}$ can be written as

$$v = \sum_{i=1}^{d} \langle v, e_i \rangle e_i.$$

Then

$$Sv = \sum_{i=1}^{d} \langle v, e_i \rangle S e_i = \sum_{i=1}^{d} \lambda_i \langle v, e_i \rangle e_i,$$

and

$$\sum_{i=1}^{n} |\langle v, f_i \rangle|^2 = \langle Sv, v \rangle = \sum_{i=1}^{d} \lambda_i |\langle v, e_i \rangle|^2.$$

Therefore,

$$\lambda_{\min} \|v\|^2 \leq \sum_{i=1}^{n} |\langle v, f_i \rangle|^2 \leq \lambda_{\max} \|v\|^2.$$

This shows that $\lambda_{\min}$ is a lower frame bound and $\lambda_{\max}$ is an upper frame bound. Taking v to be an eigenvector corresponding to $\lambda_{\min}$, respectively, $\lambda_{\max}$, proves that it is the optimal lower bound, respectively, upper bound.

(ii) We have

$$\begin{aligned} \sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \lambda_i \|e_i\|^2 &= \sum_{i=1}^{d} \langle S e_i, e_i \rangle \\ &= \sum_{i=1}^{d} \sum_{j=1}^{n} |\langle e_i, f_j \rangle|^2. \end{aligned}$$

Interchanging the order of summation and using the fact that $\{e_i\}_{i=1}^d$ is an ONB for $\mathscr{V}$ gives the desired result.

(iii) By the assumptions, $S = AI$ and so S has one eigenvalue equal to A with multiplicity d. By part (ii), this means that

$$dA = n$$

and this gives $A = \frac{n}{d}$.

□

The $n \times n$ matrix FF^* is the *Gram matrix* of the set $\{f_i\}_{i=1}^n$. The Gram matrix has rank d, and its non-zero eigenvalues are the same as the eigenvalues of the frame operator S. The $(i,j)^{\text{th}}$ entry of the Gram matrix is the inner product $\langle f_j, f_i\rangle$.

Example 3 (Types of frames).

(a) The frame $\{f_k = (\cos(2k\pi/6), \sin(2k\pi/6))\}_{k=0}^5$ given in Section 1 is a unit normed tight frame of six vectors in $\mathbb{R}^2$. See Fig. 1(a). The analysis operator is given by

$$F = \begin{bmatrix} 1 & 0 \\ 1/2 & \sqrt{3}/2 \\ -1/2 & \sqrt{3}/2 \\ -1 & 0 \\ -1/2 & -\sqrt{3}/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$$

and the frame operator is

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

We thus have a tight frame with frame bound equal to 3.

(b) The vectors $\{f_0, f_2, f_4\}$ from the frame given in (a) form a special unit normed tight frame known as the *Mercedes-Benz frame*. These vectors are the first, third and fifth rows of the matrix F. The corresponding frame operator is the matrix

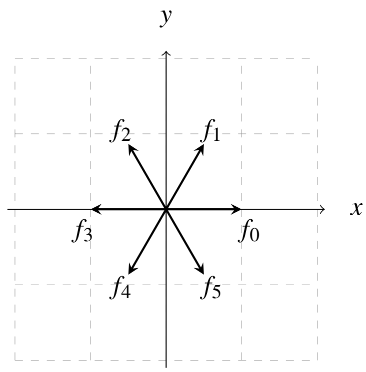
$$\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}.$$

The frame bound for this frame is therefore equal to $\frac{3}{2}$. See Fig. 1(b). This is an example of an *equiangular tight frame*, see Definition 6. Note that the absolute value of the inner product of any two distinct vectors in this set is equal to $\frac{1}{2}$.

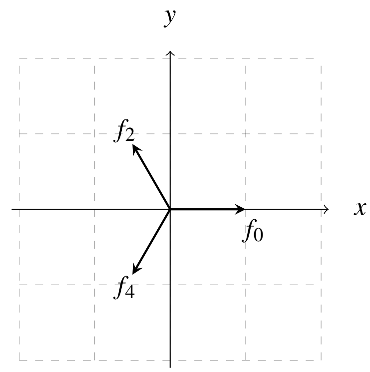
(c) Let $\mu = \frac{1}{\sqrt{5}}$. The set $\{f_k\}_{k=0}^4 \subset \mathbb{R}^3$ given by the rows of

$$\begin{bmatrix} 0 & 0 & 1 \\ \sqrt{1-\mu^2} & 0 & \mu \\ \mu\sqrt{\frac{1-\mu}{1+\mu}} & \sqrt{\frac{(1+2\mu)(1-\mu)}{1+\mu}} & \mu \\ \mu\sqrt{\frac{1-\mu}{1+\mu}} & -\sqrt{\frac{(1+2\mu)(1-\mu)}{1+\mu}} & \mu \\ -\mu\sqrt{\frac{1+\mu}{1-\mu}} & \sqrt{\frac{(1-2\mu)(1+\mu)}{1-\mu}} & \mu \end{bmatrix}$$

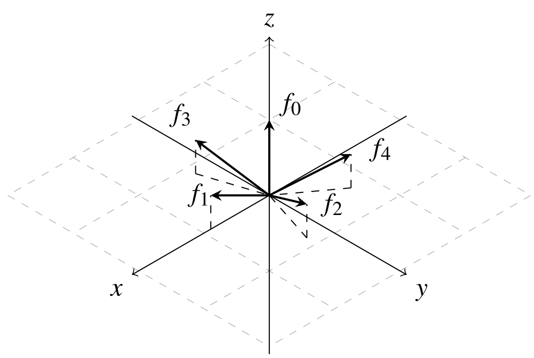
forms a unit normed frame, but not a tight frame, of five vectors in $\mathbb{R}^3$. This is an example of a *Grassmannian frame* [3], see Definition 7. The eigenvalues of the corresponding frame operator are 1 and 2. By Theorem 6, the optimal lower frame bound must be 1 and the optimal upper frame bound must be 2. See Fig. 1(c).



(a) The frame from Section 1.



(b) The Mercedes-Benz frame.



(c) The $(5,3)$ Grassmannian frame.

Fig. 1: Examples of frames.

(d) Given $n \in \mathbb{N}$, let $\omega = e^{-\frac{2\pi i}{n}}$. The $n \times n$ discrete Fourier transform (DFT) matrix is given by

$$F = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}.$$

The rows form an orthonormal basis of $\mathbb{C}^n$. Let $\widehat{F}$ denote the $n \times d$ matrix formed by selecting d columns from F. The set $\{f_k\}_{k=0}^{n-1}$ given by the rows of $\widehat{F}$ forms a tight frame in $\mathbb{C}^d$, but not necessarily a unit normed frame.

3 Equiangular Tight Frames

In a communications system, when there are several signals trying to access the same channel, it is desired that there is minimum interference between the signals. The cross correlation between two signals, given by the absolute value of their inner product, can be interpreted as a measure of their interference. This is minimized when the two vectors (signals) are mutually orthogonal i.e. their inner product is zero. However, in a finite dimensional space, the maximum number of mutually orthogonal vectors is the same as the dimension. When the number of vectors exceeds the dimension of the underlying space, we desire that the maximum cross correlation between any two vectors is minimized. Welch [24] gave the following lower bound for the maximum cross correlation among unit normed vectors $\{f_i\}_{i=1}^n$ in $\mathbb{F}^d$:

$$\max_{i\neq j} |\langle f_i, f_j\rangle| \geq \sqrt{\frac{n-d}{d(n-1)}}, \quad n \geq d. \tag{13}$$

In order to minimize the maximum cross correlation among vectors, one seeks sets of vectors that meet the lower bound of (13). Equiangular tight frames (ETFs) are a class of frames that meet the lower bound. ETFs are highly desirable and are the closest one can come to an ONB while having the redundancy of a frame, in the sense of minimizing the cross correlation while maintaining tightness. The bound $\sqrt{\frac{n-d}{d(n-1)}}$ in (13) is called the Welch bound and is denoted by α. The formal definition of an ETF is as follows.

Definition 6. [22] An equiangular tight frame (ETF) is a set $\{f_i\}_{i=1}^n$ in a d-dimensional space $\mathscr{V}$ satisfying:

(i) $F^*F = \frac{n}{d}I$, i.e., the set is a tight frame.
(ii) $\|f_i\| = 1$, for $i = 1,\ldots,n$, i.e., the set is unit normed.
(iii) $|\langle f_i, f_j\rangle| = \alpha$, $1 \leq i < j \leq n$, where α is the Welch bound.

For a given dimension d and frame size n, an ETF of n vectors in $\mathbb{F}^d$ may not exist [22]. Even when they do exist, ETFs are hard to construct. However, ETFs of $d+1$ vectors for dimension d always exist and can be viewed as the vertices of a regular simplex centered at the origin [21, 22]. Examples of such ETFs can be found in [13], and an explicit construction is also given in [8]. Note that for such an ETF the Welch bound is given by $\alpha = \frac{1}{d}$. An example of an ETF of three vectors in $\mathbb{R}^2$ was given earlier in Example 3(b).

Example 4 (An ETF of four vectors in $\mathbb{R}^3$). Let $d = 3$. Consider the four vectors given by

$$f_1 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ -\frac{\sqrt{2}}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad f_2 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{\sqrt{2}}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 \\ \frac{2\sqrt{2}}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad f_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In this case, for $i \neq j$, $\langle f_i, f_j \rangle = -1/3$.

When ETFs cannot exist, the Welch bound cannot be attained by any set of n vectors in $\mathbb{F}^d$. However, the maximum cross correlation between frame vectors can still be minimized even if the minimum does not coincide with the Welch bound. Such sets are called *Grassmannian frames* [3].

Definition 7. A frame of n vectors in $\mathbb{F}^d$ is called a Grassmannian frame if it is a solution to

$$\min\{\max_{i \neq j} |\langle f_i, f_j \rangle|\}$$

where the minimum is taken over all unit normed frames $\{f_i\}_{i=1}^n$ in $\mathbb{F}^d$.

An example of a Grassmannian frame of five vectors in $\mathbb{R}^3$ is shown in Fig. 1(c).

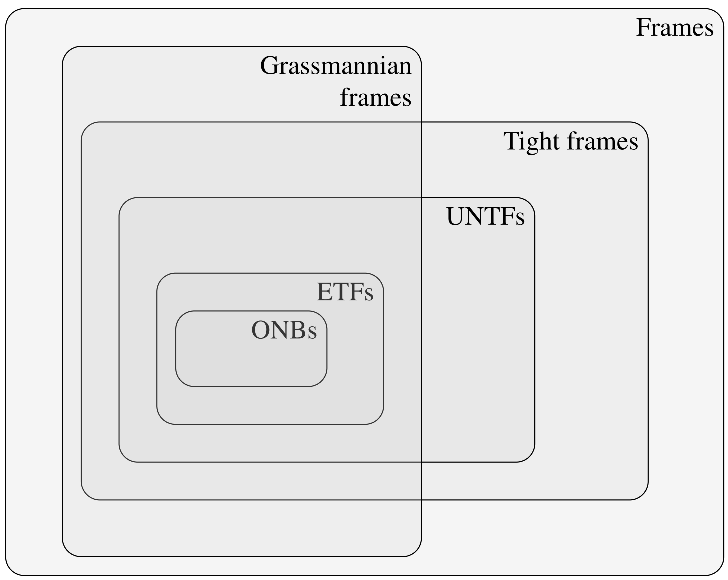


Fig. 2: Families of frames.

3.1 *k-angle Tight Frames*

As mentioned previously, equiangular tight frames are useful in applications because they minimize the maximum cross correlation among pairs of unit vectors, but they are also rare. Therefore, it is desirable to construct sets that mimic ETFs in some way. If $\{f_i\}_{i=1}^n$ is an ETF in $\mathbb{R}^d$ or $\mathbb{C}^d$, then when $i \neq j$, $|\langle f_i, f_j \rangle|$ must be equal to the Welch bound α. Relaxing this condition, one obtains a larger class of frames that contains the equiangular tight frames. In particular, a *k-angle tight frame* is a set $\{f_i\}_{i=1}^n$ in a d-dimensional space $\mathscr{V}$ satisfying [8]:

(i) $F^*F = \frac{n}{d}I$, i.e., the set is a tight frame.
(ii) $\|f_i\| = 1$ for $i = 1, \ldots, n$, i.e., the set is unit normed.
(iii) $|\langle f_i, f_j \rangle| \in \{\alpha_m\}_{m=1}^k$ for $1 \leq i < j \leq n$, where $\{\alpha_m\}_{m=1}^k \subset [0,1]$.

Example 5 (Mutually unbiased bases). Two orthonormal bases $\{f_i\}_{i=1}^d$ and $\{g_i\}_{i=1}^d$ in a d-dimensional space $\mathscr{V}$ form a pair of *mutually unbiased bases* if $|\langle f_i, g_j \rangle| = \frac{1}{\sqrt{d}}$ for $1 \leq i, j \leq d$. Let $\{\widetilde{f}_i\}_{i=1}^{2d}$ denote the union $\{f_i\}_{i=1}^d \cup \{g_i\}_{i=1}^d$. Then $\{\widetilde{f}_i\}_{i=1}^{2d}$ is a 2-angle tight frame, since $|\langle \widetilde{f}_i, \widetilde{f}_j \rangle| \in \{0, \frac{1}{\sqrt{d}}\}$ for $1 \leq i < j \leq 2d$.

A way to construct k-angle tight frames uses ETFs as starting points.

Theorem 7. *[8] Let $d, k \in \mathbb{N}$ with $k < d+1$, and set $d' = \binom{d+1}{k}$. Denote the collection of all subsets of $\{1, \ldots, d+1\}$ of size k by $\{\Lambda_i\}_{i=1}^{d'}$. Let $\{f_i\}_{i=1}^{d+1} \subseteq \mathbb{R}^d$ denote the ETF with $\langle f_i, f_j \rangle = -\frac{1}{d}$ for $i \neq j$. Define a new collection $\{g_i\}_{i=1}^{d'}$ as follows:*

$$g_i := \frac{\sum_{j \in \Lambda_i} f_j}{\|\sum_{j \in \Lambda_i} f_j\|}.$$

Then $\{g_i\}_{i=1}^{d'}$ forms a $\hat{k}$-angle tight frame of d' vectors in $\mathbb{R}^d$, where $\hat{k} \leq k$.

The proof of Theorem 7 can be found in [8]. The construction of the starting ETF with $\langle f_i, f_j \rangle = -\frac{1}{d}$, $i \neq j$, can be done based on an algorithm in [8], and an example of such an ETF for $d = 3$ has been provided earlier in this section in Example 4. The main idea behind the proof of the tightness part is to compute the frame potential of $\{g_i\}_{i=1}^{d'}$ and then use Theorem 5.

Example 6 (A 2-angle tight frame in $\mathbb{R}^2$). Consider the Mercedes-Benz (MB) frame discussed in Example 3(b), and shown below in Fig. 3. It consists of the three vectors in $\mathbb{R}^2$ given by

$$f_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ f_1 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \ f_2 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}.$$

Note that for $i \neq j$, $\langle f_i, f_j \rangle = -1/2$. Let $k = 2$. Follow the construction given in Theorem 7, i.e., for every distinct pair f_i, f_j, $i < j$, in the MB frame, calculate $\frac{f_i + f_j}{\|f_i + f_j\|}$ to get three more vectors. Now add these three vectors to the MB frame. Doing so produces a unit normed tight frame of six vectors in $\mathbb{R}^2$, that was shown in Fig. 1(a), and is reproduced in Fig. 4. $\{f_i\}_{i=0}^5$ is a 2-angle tight frame: for $0 \leq i < j \leq 5$, $\langle f_i, f_j \rangle \in \{-\frac{1}{3}, \frac{1}{3}, -1\}$ which means that $|\langle f_i, f_j \rangle| \in \{\frac{1}{3}, 1\}$.

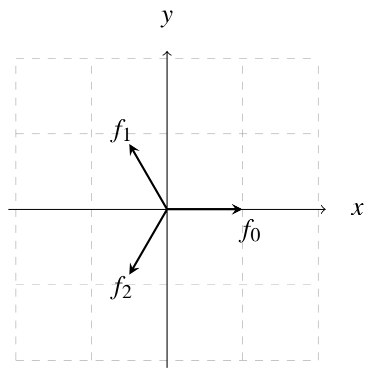


Fig. 3: The Mercedes-Benz frame.

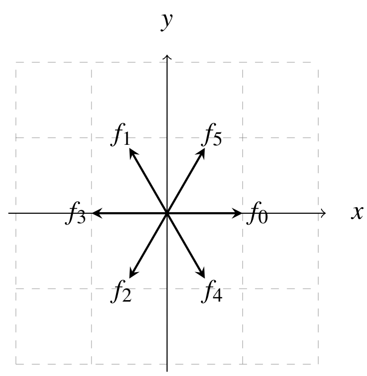


Fig. 4: A 2-angle tight frame in $\mathbb{R}^2$.

3.2 Tight Frames and Graphs

Equiangular tight frames have important connections to graph theory. Perhaps the most well-known connection is that between ETFs and graph theoretic objects known as *regular two-graphs* [16, 21]. In particular, [16] gives a one-to-one correspondence between equivalence classes of real ETFs and regular two-graphs. Another important connection exists between ETFs and strongly regular graphs as described below. A graph for which every vertex has the same number of neighbors is a *regular graph*. A regular graph $\mathscr{G}$ is called a *strongly regular graph* if any two adjacent vertices in $\mathscr{G}$ have λ common neighbors, and any two non-adjacent vertices in $\mathscr{G}$ have μ common neighbors. In [23], it is shown that a real ETF $\{f_i\}_{i=1}^n$ in $\mathbb{R}^d$ exists if and only if a particular strongly regular graph exists. Graph theory has also proven useful in obtaining error estimates for signal reconstruction when using ETFs in the presence of transmission losses [4].

These connections go beyond characterizing ETFs. A unit normed tight frame $\{f_i\}_{i=1}^n \subset \mathbb{R}^d$ is called a *two-distance tight frame*[3] if $\langle f_i, f_j \rangle \in \{\alpha_1, \alpha_2\} \subset [-1,1]$ for $\alpha_1 \neq \alpha_2$ and $1 \leq i < j \leq n$. If G is the Gram matrix of a two-distance tight frame, then $G = I + \alpha_1 Q_1 + \alpha_2 Q_2$, where Q_1 and Q_2 are symmetric binary matrices with zeros on the diagonal. Recall that a graph $\mathscr{G}$ has adjacency matrix $Q = [q_{ij}]$ defined by

$$q_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

The matrices Q_1 and Q_2 in the decomposition of the Gram matrix G above can then be viewed as adjacency matrices of graphs. In [1], the authors prove the following result.

Theorem 8 (Proposition 3.2, [1]). *Let $\{f_i\}_{i=1}^n$ be a two-distance tight frame with Gram matrix $G = I + \alpha_1 Q_1 + \alpha_2 Q_2$, where $\alpha_1 \neq \pm\alpha_2$. Then Q_1 and Q_2 are both adjacency matrices of strongly regular graphs.*

4 Possible Research Topics

4.1 Construction of Grassmannian Frames

Equiangular tight frames are important in many applications, but as discussed previously they do not always exist. However, ETFs are themselves a specific example of a larger class of frames called Grassmannian frames (see Fig. 2), and for any choice of d, n, a Grassmannian frame always exists [3]. Grassmannian frames are important in frame theory for many of the same reasons that ETFs are, since Grassmannian frames minimize the maximum cross correlation among sets of unit vectors.

Despite their importance, many questions remain unanswered about the construction of Grassmannian frames for given values of d, n, in either $\mathbb{R}^d$ or $\mathbb{C}^d$. Constructions in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been given in [3].

Research Project 1. Some interesting questions that can be explored in the context of Grassmannian frames are the following.

- How can one verify whether or not a given frame is a Grassmannian frame?
- When can one construct Grassmannian frames from previously existing "structured" objects such as ETFs?
- Improvements on the Welch bound exist in certain situations, such as the following bound given in [25]:

[3] Note that two-distance tight frames as defined here and in [1] are either ETFs or 2-angle tight frames from the beginning of Subsection 3.1.

$$\max_{i \neq j} |\langle f_i, f_j \rangle| \geq \max \left\{ \sqrt{\frac{n-d}{d(n-1)}}, 1 - 2n^{-\frac{1}{d-1}} \right\}.$$

How sharp are these bounds for various values of d and n? Using computer experiments to find a wide variety of unit normed frames and comparing their maximum cross correlation with these bounds could give insight into how close to the lower bound one can get and what the minimizers of the maximum cross correlation could be when ETFs do not exist.

4.2 *k*-angle Tight Frames and Regular Graphs

As mentioned in Subsection 3.2, there exists a correspondence between certain 2-angle tight frames and strongly regular graphs. A natural course of action would be to extend this result to k-angle tight frames where $k \geq 3$, and some progress has already been made in the case $k = 3$ [19]. Let G be the Gram matrix of a k-angle tight frame $\{f_i\}$, and suppose that $|\langle f_{i_1}, f_{j_1} \rangle| = |\langle f_{i_2}, f_{j_2} \rangle|$ if and only if $\langle f_{i_1}, f_{j_1} \rangle = \langle f_{i_2}, f_{j_2} \rangle$. In other words, if $\alpha \in \{\langle f_i, f_j \rangle\}$, then $-\alpha \notin \{\langle f_i, f_j \rangle\}$. Let $\{\alpha_i\}_{i=1}^k = \{\langle f_i, f_j \rangle\}$, which are the distinct off-diagonal entries of G. Then

$$G = I + \alpha_1 Q_1 + \cdots + \alpha_k Q_k, \tag{14}$$

where the matrices Q_i are symmetric binary matrices with 0s along the diagonal for $1 \leq i \leq k$. Each matrix Q_i is the adjacency matrix of a graph. The question then is as follows:

Research Project 2.
What can be said about the structure of the graphs determined by the matrices $\{Q_i\}_{i=1}^k$ in (14)?

To illustrate this question, consider the matrix G given by

$$G=\begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{1}{6} & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & 1 & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 1 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & 1 & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 1 & \frac{1}{6} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 1 \end{bmatrix}$$

G is the Gram matrix of a 2-angle tight frame in $\mathbb{R}^4$. If Q_1 and Q_2 are defined as

$$Q_1=\begin{bmatrix} 0&0&0&0&0&1&0&0&1&1 \\ 0&0&0&0&1&0&0&1&0&1 \\ 0&0&0&1&0&0&1&0&0&1 \\ 0&0&1&0&0&0&0&1&1&0 \\ 0&1&0&0&0&0&1&0&1&0 \\ 1&0&0&0&0&0&1&1&0&0 \\ 0&0&1&0&1&1&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0&0 \\ 1&0&0&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0 \end{bmatrix} \quad \text{and} \quad Q_2=\begin{bmatrix} 0&1&1&1&1&0&1&1&0&0 \\ 1&0&1&1&0&1&1&0&1&0 \\ 1&1&0&0&1&1&0&1&1&0 \\ 1&1&0&0&1&1&1&0&0&1 \\ 1&0&1&1&0&1&0&1&0&1 \\ 0&1&1&1&1&0&0&0&1&1 \\ 1&1&0&1&0&0&0&1&1&1 \\ 1&0&1&0&1&0&1&0&1&1 \\ 0&1&1&0&0&1&1&1&0&1 \\ 0&0&0&1&1&1&1&1&1&0 \end{bmatrix},$$

then

$$G=I-\frac{2}{3}Q_1+\frac{1}{6}Q_2.$$

Since each row of Q_1 contains three 1s, this means that every vertex of the graph whose adjacency matrix is Q_1 has three adjacent vertices, hence this graph is regular. A similar statement is true for Q_2 and the graph associated with it. See Fig. 5. In fact, the graphs associated to Q_1 and Q_2 are strongly regular as well by Theorem 8, but the purpose of this example is only to illustrate how one obtains graphs from k-angle tight frames and studies their structure.

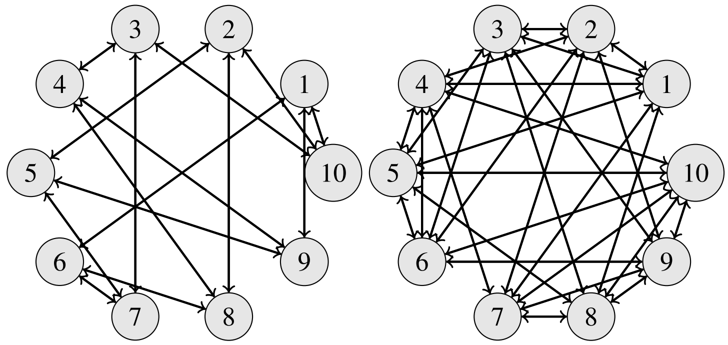


Fig. 5: The graphs corresponding to Q_1 and Q_2.

4.3 Frame Design Issues

Designing frames with good properties is important. This chapter has discussed an important class of frames called equiangular tight frames. When designing frames, it is important to know the kinds of transformations under which a given frame does not lose the properties it already possesses. For example, if U is a unitary transformation and if $\{f_i\}$ is a unit normed tight frame, then the set $\{Uf_i\}$ is also a unit normed tight frame. If $\{f_i\}$ is an ETF, then $\{Uf_i\}$ is also an ETF. Recall that for a tight frame the frame operator S has a nice structure, being a constant multiple of the identity. In other words, S is a diagonal matrix with all diagonal entries equal to A, the frame bound. Having such a structure is useful for computational reasons. In particular, it is convenient to invert S in the expansion formula (12), and the process is numerically stable. One can ask the following.

Research Project 3. What are the operators M for which $\{Mf_i\}$ is a frame whose frame operator S is a diagonal matrix?

This is useful since diagonal matrices are also easy to invert and the frame $\{Mf_i\}$ can therefore offer advantages similar to that of a tight frame. Of course, there are other variations of the above question that can be considered.

In the context of erasures or losses, as discussed in Section 1, suppose that e number of frame coefficients are lost during transmission. If the index set of erasures is denoted by E, then to the receiver it is as if the signal is to be recovered using the frame $\{f_i\}_{i\notin E}$, where $|E| = e$, assuming that $\{f_i\}_{i\notin E}$ is a frame. In the presence of noise, it has been shown [13] that when starting with a unit normed frame the mean-squared error is minimized if the remaining vectors $\{f_i\}_{i\notin E}$ form a tight frame. Unfortunately, as given in Theorem 4.3 in [13], it is not possible for *every* $\{f_i\}_{i\notin E}$

with $|E| = e$ to be tight. In this context, it might be interesting to investigate the following.

Research Project 4. Fix the number of erasures e such that $e \leq n-d$, where n represents the size of a frame and d is the dimension. Starting with a unit normed frame of n vectors in $\mathbb{F}^d$, remove e vectors and see for how many of the $\binom{n}{e}$ cases the set $\{f_i\}_{i \notin E}$ is a tight frame. Try this for different unit normed frames using a computer. Is it possible to characterize the starting frames that maximize the number of cases when $\{f_i\}_{i \notin E}$ is a tight frame?

4.4 Frame Algorithms

In order to reconstruct a signal v from the frame coefficients $\{\langle v, f_i \rangle\}_{i=1}^n$ one can use (12). However, that entails inverting the frame operator which can be complicated if the dimension is large. To avoid inverting S one can find successive approximations to f. This can be done using a well-known algorithm known as the *frame algorithm* [6]. This is given below in Lemma 4.

Lemma 4. *[6] Let $\{f_k\}_{k=1}^n$ be a frame for $\mathscr{V}$ with frame bounds A, B. Given $v \in \mathscr{V}$, define the sequence $\{g_k\}_{k=0}^\infty$ in $\mathscr{V}$ by*

$$g_0 = 0,\ g_k = g_{k-1} + \frac{2}{A+B} S(f - g_{k-1}), k \geq 1. \tag{15}$$

Then

$$\|f - g_k\| \leq \left(\frac{B-A}{B+A} \right)^k \|f\|.$$

It should be noted in Lemma 4 above that if B is much larger than A then the convergence is slow. There are acceleration algorithms based on the Chebyshev method and the conjugate gradient method [14] that improve the speed of convergence in (15). Other algorithms on approximating the inverse frame operator include work in [20, 5]. An interesting project might be the following.

Research Project 5. Compare the various existing algorithms for inverting a frame operator and study other techniques to improve existing methods.

4.5 Reconstruction in Presence of Random Noise

Assume that during transmission the frame coefficients get corrupted by some random noise so that what is received are the corrupted coefficients $\{\langle v, f_i\rangle + \eta_i\}_{i=1}^n$ where each η_i has mean zero and variance σ^2. Further, for $i \neq j$, η_i and η_j are uncorrelated. The reconstruction of the signal x from the noisy coefficients is done as follows:

$$\widehat{x} = \sum_{i=1}^{n} [\langle v, f_i\rangle + \eta_i] S^{-1} f_i = x + \sum_{i=1}^{n} \eta_i S^{-1} f_i.$$

Due to the assumptions,

$$E[\widehat{x}] = x.$$

Assuming an unbiased estimator, the mean-squared error (MSE) is the trace of the covariance matrix of $\widehat{x}$. It has been shown in [13] that for a single erasure (or lost coefficient), among all unit normed frames, tight frames minimize the MSE. It is also shown in [13] that under a single erasure from a unit normed frame, the MSE averaged over all erasures is minimized if and only if the starting frame is tight. In [13], a uniform distribution is taken for the erasure model, i.e., each coefficient is equally likely to be lost. This leads to the following.

Research Project 6. Study the effect of a general distribution for the erasure model, and investigate the properties of the starting frame that would minimize the MSE. As one example, how well do k-angle tight frames, as described in Subsection 3.1, perform in the face of erasures?

For more than two erasures, it is known that all tight frames do not minimize the two-erasure MSE. It has been shown in [16] that under two erasures the optimal frames are ones that are equiangular. So far, it has been assumed that when a frame vector has been erased the remaining vectors still form a frame. In this case, assuming that the location of the loss is known, the signal recovery is done using the frame operator of the new frame. However, it may be the case that eliminating just a single vector leaves a set that is no longer a frame. This consideration can lead to other areas of research.

It might also be interesting to investigate the effect of random noise or other random phenomena on equiangularity of an ETF.

Research Project 7. Starting from an ETF, estimate the deviations of the frame from being equiangular or being tight by adding random perturbations to the frame vectors.

A preliminary analysis of this is done in [7].

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