Welch bound equality sets with few distinct inner products from Delsarte-Goethals sets

Somantika Datta[1*]

[1*]Department of Mathematics and Statistical Science, University of Idaho, 875 Perimeter Drive, Idaho 83843-1103, USA.

Corresponding author(s). E-mail(s): sdatta@uidaho.edu;

Abstract

Sets of signals that meet Welch bounds with equality or near equality are of value in communications and sensing applications, and the construction of such signal sets has been an active research area. Although Welch derived a family of bounds indexed by positive integers $\boldsymbol{k}$, only the first Welch bound (i.e., for $\boldsymbol{k=1}$) has been considered in these constructions. Earlier, a frame-theoretic perspective was introduced on the higher Welch bounds that is valuable in constructing signals that simultaneously meet multiple Welch bounds with equality or near equality. This perspective is used in this paper to examine the existence of signal sets that meet the $\boldsymbol{k}^{\text{th}}$ Welch bound with equality by using second order Reed-Muller codes. Some examples of such signal sets are presented and connections to equiangular lines and $\boldsymbol{t}$-designs are discussed.

Keywords: Delsarte-Goethals sets, Reed-Muller codes, mutually unbiased bases, t-designs, tight frames, Welch bound

1 Introduction

Welch's bounds [1] apply to sets $X = \{x_1, \ldots, x_m\}$ of unit-norm vectors in $\mathbb{C}^n$. Denoting by $\langle x_i, x_j \rangle$ the inner product of x_i and x_j, the fundamental bound in [1] gives

$$\sum_{i=1}^{m}\sum_{j=1}^{m} | \langle x_i, x_j \rangle |^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}} \tag{1}$$

for each integer $k \geq 1$. Since the x_i have unit norm, (1) is equivalent to

$$\sum_{i \neq j} | \langle x_i, x_j \rangle |^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}} - m. \tag{2}$$

It can be shown that the inequality is satisfied with equality if and only if X is a tight frame for $\mathbb{C}^n$ with bound $B = m/n$ (see, e.g., [2–4]).

Continuing with the case $k = 1$, Welch also noted that, since terms in the sum on the left side of (2) are non-negative, the sum must be at least as large as the number of terms times the largest term; i.e.,

$$c_{\max}^2 := \max_{i \neq j} | \langle x_i, x_j \rangle |^2 \geqslant \frac{1}{m-1} \left[\frac{m}{n} - 1 \right]. \tag{3}$$

Meeting this bound with equality requires not only that X is a tight frame for $\mathbb{C}^n$, but that all the inner products $\langle x_i, x_j \rangle$ with $i \neq j$ are equal in magnitude; i.e, that X is a set of equiangular lines in $\mathbb{C}^n$ [3, Theorem 2.3].

A signal set $X \subset \mathbb{C}^n$ that meets inequality (1) with equality is known as a *Welch Bound Equality (WBE)* set [5–7]. In the literature, WBE sets are also referred to as *Grassmannian frames* [3]. X meets inequality (3) with equality if and only if X is an equiangular tight frame (ETF) [3, Theorem 2.3]. Such sets are called *Maximal Welch Bound Equality (MWBE)* sets [5–7] or *optimal Grassmannian frames* [3], or *two-uniform frames* [8]. Sets attaining the bound in (1) or (3) with equality are of value in communications and active sensing applications, and construction of such sets for the $k = 1$ case has been an active research area [3, 5–7, 9].

In [10], the authors formulated a geometric perspective on the higher Welch bounds; i.e., (1) and the analogue of (3) with $k \geq 2$ which takes the form

$$c_{max}^{2k} \geqslant \frac{1}{m-1} \left[\frac{m}{\binom{n+k-1}{k}} - 1 \right]. \tag{4}$$

The central goal of this paper is to exploit this perspective to examine the existence and construction of signal sets in $\mathbb{C}^n$ that meet these higher bounds with equality. To seek such sets, we study second order Reed-Muller codes and special subspaces of these codes, namely, the Kerdock and Delsarte-Goethals sets. Our investigation yields results and examples for $k \leq 3$; obtaining sets with equality for $k > 2$ is a challenging problem. We will refer to a set that meets inequality (1) with equality as a WBE set whereas a set that meets (3) with equality will be referred to as an ETF. For $k > 1$, a set that meets (4) with equality will be called a k-ETF. In situations where $k \geq 1$, 1-ETF will be the usual meaning of an ETF.

The geometric perspective used to formulate the higher order Welch bounds in [10] relies on the space of symmetric tensors, $\mathrm{Sym}^k(\mathbb{C}^n)$. The dimension of

$\mathrm{Sym}^k(\mathbb{C}^n)$ is denoted by $d(n,k)$. Note that

$$d(n,k) = \binom{n+k-1}{k},$$

and the inner product in $\mathrm{Sym}^k(\mathbb{C}^n)$ is defined by

$$\langle x_i^{\otimes k}, x_j^{\otimes k}\rangle := \langle x_i, x_j\rangle^k.$$

In [10, Theorem 3.1] it is shown that the set $X^{\otimes k} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ satisfies (1) with equality if and only if it is a tight frame for $\mathrm{Sym}^k(\mathbb{C}^n)$ and satisfies (4) with equality if and only if $X^{\otimes k}$ is an equiangular tight frame of $\mathrm{Sym}^k(\mathbb{C}^n)$. Combining this with a result in [2, Theorem 1] gives

Theorem 1 *The following are equivalent:*

1. *X is a complex projective t-design in $\mathbb{C}^n$.*
2. *$X^{\otimes k}$ is a WBE set for $k = 1, \ldots, t$.*
3. *$X^{\otimes k}$ is a tight frame for $\mathrm{Sym}^k(\mathbb{C}^n)$ for $k = 1, \ldots, t$ with frame bound $\frac{m}{d(n,k)}$.*

2 k-ETFs

Suppose X is an ETF in $\mathbb{C}^n$. Certain necessary conditions for a set to be a k-ETF for $k > 1$, i.e., attain equality in (4), can be obtained purely from dimensionality conditions. This is given below in Theorem 2.

Theorem 2 *Let $X = \{x_1, \ldots, x_m\}$ be an ETF in $\mathbb{C}^n$. Then*
(a) $X^{\otimes k}$ is a set of equiangular lines in $\mathrm{Sym}^k(\mathbb{C}^n)$ for all $k > 1$.
(b) If $X^{\otimes k}$ is a k-ETF for $k > 1$, then $d(n,k) \leq m \leq n^2$ where $d(n,k)$ is the dimension of $\mathrm{Sym}^k(\mathbb{C}^n)$.

Proof (a) Let $G_X = [\langle x_i, x_j\rangle]_{1\leq i,j\leq m}$ be the Gram matrix of X. Then G_X will have ones on the main diagonal and all off-diagonal entries will be of equal modulus; i.e., $|\left\langle x_i, x_j\right\rangle| = \alpha$ for all $i \neq j$. The Gram matrix associated with $X^{\otimes k}$ in $\mathrm{Sym}^k(\mathbb{C}^n)$ is

$$G_{X^{\otimes k}} = \begin{bmatrix} 1 & \cdots & \left\langle x_1^{\otimes k}, x_m^{\otimes k}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_m^{\otimes k}, x_1^{\otimes k}\right\rangle & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1^k & \cdots & \langle x_1, x_m\rangle^k \\ \vdots & \ddots & \vdots \\ \langle x_m, x_1\rangle^k & \cdots & 1^k \end{bmatrix}.$$

Hence $G_{X^{\otimes k}}$ has ones on the main diagonal and all off-diagonal entries of modulus α^k. So $X^{\otimes k}$ is a set of equiangular lines in $\mathrm{Sym}^k(\mathbb{C}^n)$.
(b) If $X^{\otimes k}$ is a k-ETF for $k > 1$, then $X^{\otimes k}$ is an equiangular tight frame of $\mathrm{Sym}^k(\mathbb{C}^n)$ [10, Theorem 3.1]. In order to be a frame, it must span $\mathrm{Sym}^k(\mathbb{C}^n)$. Thus, based on dimensionality, it is necessary that $m \geq d(n,k)$.

Table 1 Feasibility of k and dimension n for k-ETFs by Theorem 2

n / k	2	3	4	...
1	✓	✓	✓	✓
2	✓	✓	✓	✓
3	✓	X	X	X
4	X	X	X	X

Recall that the maximal number of equiangular vectors in $\mathbb{C}^n$ cannot exceed n^2 ([11, 12]). Thus, to obtain k-ETFs by taking tensor products of a certain equiangular tight frame X in $\mathbb{C}^n$, it is necessary that $m \leq n^2$.

□

Remark 1 In Theorem 2 (b), the dimensionality condition is not sufficient for being a k-ETF for the following reason. If $m \geq d(n,k)$ and the rank of $G_{X^{\otimes k}}$ is $d(n,k)$, then $X^{\otimes k}$ is a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. Part (a) shows that $X^{\otimes k}$ is an equiangular set. However, this frame may not be tight. Even if X is a tight frame for $\mathbb{C}^n$, $X^{\otimes k}$ need not be a tight frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. There are techniques to get a tight frame from a frame. For example, the set $\{(\mathcal{F}^{(k)})^{-\frac{1}{2}} x_j^{\otimes k}\}_{j=1}^m$, where $\mathcal{F}^{(k)}$ is the frame operator[1] of $X^{\otimes k}$, is a tight frame in $\mathrm{Sym}^k(\mathbb{C}^n)$; however, the equiangularity of $X^{\otimes k}$ is not preserved by this operation.

Example 1 For a given n, Theorem 2 requires that

$$d(n,k) = \binom{n+k-1}{k} \leq n^2.$$

Take $n = 2$. Then

$$d(2,k) = \binom{2+k-1}{k} = k+1 \leq 4,$$

i.e., $k \leq 3$ must hold. When $n = 3$, it is required that

$$d(3,k) = \binom{3+k-1}{k} = \frac{(k+1)(k+2)}{2} \leq 9.$$

However, there is no $k > 2$ for which this is satisfied. Similarly, for $n = 4$, the condition $d(n,k) \leq n^2$ implies $\frac{(k+1)(k+2)(k+3)}{6} \leq 16$, and there is no $k > 2$ for which this holds. For any $n > 4$, writing $d(n,k)$ as

$$d(n,k) = \binom{n+k-1}{k} = n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdots \frac{n+k-1}{k},$$

it can be seen that $d(n,k) \leq n^2$ is satisfied only for $k = 1$ and $k = 2$. The feasibility of some pairs (k,n) by Theorem 2 (b), as calculated above, is shown in Table 1.

[1] Given a finite frame $X = \{x_1, \ldots, x_m\}$ for an n-dimensional complex vector space V, the function $F : V \to \mathbb{C}^m$ given by $F(w) = [\langle x_1, w\rangle \ \ldots \ \langle x_m, w\rangle]^{\mathrm{T}}$ will be called the *analysis operator* associated with X, while $\mathcal{F} = F^*F : V \to V$ (i.e., the composition of the adjoint of F with F) will be called the *frame operator* associated with X.

Remark 2 For $k > 1$, Theorem 2 depends on a starting 1-ETF, i.e., an ETF of m vectors in $\mathbb{C}^n$. Pairs (m, n) for which ETFs of m vectors can exist in $\mathbb{C}^n$, and the required conditions along with examples are discussed in [8, 13, 14] from where it is also known that ETFs are rare. One should note that due to the inner product property of $\mathrm{Sym}^k(\mathbb{C}^n)$, if ETFs do not exist for a certain pair (m, n) then k-ETFs of such cardinalities m also cannot exist for $k > 1$. Example 1 shows that even if there is a 1-ETF X, one cannot reasonably extend this to k-ETFs for k's greater than 2 by taking $X^{\otimes k}$. These are some of the issues faced in the construction of k-ETFs or sets that meet the higher bounds in (4).

Example 2 (A k-ETF for $k = 1, 2$) Maximal equiangular sets in $\mathbb{C}^n$ of n^2 unit vectors satisfying the Welch bound (3), i.e., $|\langle x_i, x_j\rangle|^2 = \frac{1}{n+1}$, $i \neq j$, are called symmetric informationally complete positive operator valued measures (SIC-POVMs). This means that SIC-POVMs are ETFs. SIC-POVMs are conjectured to exist for all n. The analytical proof for the existence of SIC-POVMs in any arbitrary dimension remains an open problem. Analytic solutions for the existence of SIC-POVMs exist for some dimensions [15–18]. Numerical solutions of SIC-POVMs for many dimensions are currently available [15, 17, 19]. It has been shown in [15, Theorem 2] and [2, Theorem 5] that SIC-POVMs form complex projective 2-designs. By Theorem 1, these are WBE sets for $k = 1, 2$. Further, due to the equiangularity of SIC-POVMs, these form k-ETFs for $k = 1, 2$. From the inner product property of these sets, this can also be shown through a direct calculation by showing that SIC-POVMs satisfy (4) for $k = 1$ and $k = 2$, and hence form k-ETFs for $k = 1, 2$.

3 Higher WBE sets using Delsarte-Goethals sets

Consider the space of ℓ dimensional binary vectors $\mathbb{Z}_2^\ell$, also called the *Hamming space*. The second order Reed-Muller code of length 2^ℓ is parameterized by $\ell \times \ell$ binary symmetric matrices P and binary vectors $b \in \mathbb{Z}_2^\ell$. Let d_P denote the main diagonal of a matrix P, and by $\mathrm{wt}(d_P)$ we will mean the number of ones in the binary vector d_P. In terms of the parameters P and b, a second order Reed-Muller codeword is given by

$$\phi_{P,b}(a) = \frac{1}{\sqrt{2^\ell}} i^{\mathrm{wt}(d_P)+2\mathrm{wt}(b)} i^{(2b+Pa)^T a} \tag{5}$$

where $i = \sqrt{-1}$. In the above expression (5), $a \in \mathbb{Z}_2^\ell$ indexes the 2^ℓ components of the codeword $\phi_{P,b}$. For a fixed binary symmetric matrix P, the set

$$\mathcal{F}_P = \{\phi_{P,b} : b \in \mathbb{Z}_2^\ell\}$$

forms an orthonormal basis for $\mathbb{C}^{2^\ell}$. For a fixed ℓ, the total number of symmetric matrices is $2^{\ell(\ell+1)/2}$. These give a set of $2^{\ell(\ell+1)/2}$ orthonormal bases for $\mathbb{C}^{2^\ell}$. For each P, the vectors of the set $\mathcal{F}_P$ can be used as rows (or columns) to form a $2^\ell \times 2^\ell$ unitary matrix U_P. By concatenating the matrices U_{P_i},

$i = 1, \dots, 2^{\ell(\ell+1)/2}$, one obtains the $2^\ell \times 2^{\ell(\ell+3)/2}$ matrix

$$\Phi = \left[U_{P_1} \ U_{P_2} \ \dots \ U_{P_{2^{\ell(\ell+1)/2}}} \right] \tag{6}$$

whose columns are the Reed-Muller codewords of length 2^ℓ. The following Lemma 3 follows directly from Lemma 7 and Theorem 1.

Lemma 3 *(i) The columns of Φ form a WBE set of $2^{\ell(\ell+3)/2}$ vectors in $\mathbb{C}^{2^\ell}$ for $k = 1$.*
(ii) Let N be an integer such that $1 \le N \le 2^{\ell(\ell+1)/2}$. Then $\cup_{n=1}^{N} U_{P_{i_n}}$, where $i_n \in \{1, 2, \dots, 2^{\ell(\ell+1)/2}\}$, is a WBE set of $N2^\ell$ vectors in $\mathbb{C}^{2^\ell}$ for $k = 1$.

Let ℓ be an odd number. The Delsarte-Goethals set $DG(\ell, h)$ is a binary vector space containing $2^{\ell(h+1)}$ binary symmetric matrices of size $\ell \times \ell$ with the property that for any distinct pair P, Q in $DG(\ell, h)$, the rank of the binary sum $P + Q$ is at least $\ell - 2h$ [20, 21]. The Delsarte-Goethals sets are nested

$$DG(\ell, 0) \subset DG(\ell, 1) \subset \cdots \subset D(\ell, \frac{\ell - 1}{2}).$$

The set $DG(\ell, 0)$ is the classical *Kerdock set* [22]. The size of the Kerdock set is 2^ℓ. The last set $DG(\ell, \frac{\ell-1}{2})$ is the set of all binary symmetric $\ell \times \ell$ matrices and this gives the entire second order Reed-Muller code.

Due to orthonormality, the inner product between two different columns of the same matrix U_{P_i} is zero. Fixing a vector v_i which is the column of a certain matrix U_{P_i} and letting another vector v_j range over the columns of a different matrix U_{P_j}, the inner product satisfies [21, 23]

$$| \langle v_i, v_j \rangle | = \begin{cases} \frac{1}{\sqrt{2^r}}, & 2^r \text{ times}, \\ 0, & 2^\ell - 2^r \text{ times}, \end{cases} \tag{7}$$

where $r = \text{rank}(P_i - P_j)$.
Theorem 4 below shows that the Kerdock set gives rise to WBE sets for $k = 1, 2$. The following definition is needed in Theorem 4 .

Definition 1 Two orthonormal bases B and B' of $\mathbb{C}^n$ are said to be *mutually unbiased* when $| \langle b, b' \rangle |^2 = 1/n$ holds for all $b \in B$ and $b' \in B'$.

Theorem 4 *For $P_{i_k} \in DG(\ell, 0)$, $1 \le k \le 2^\ell$, consider*

$$\Phi_{Ker} = \left[U_{P_{i_1}} \ U_{P_{i_2}} \ \cdots \ U_{P_{i_{2^\ell}}} \ I_{2^\ell} \right]$$

where I_{2^ℓ} is the $2^\ell \times 2^\ell$ identity matrix. For $k = 1, 2$ in (1), the columns of Φ_{Ker} form a WBE set of $2^\ell(2^\ell + 1)$ vectors in $\mathbb{C}^{2^\ell}$, and $\Phi_{Ker}^{\otimes 2}$ is a tight frame for $Sym^2(\mathbb{C}^{2^\ell})$. However, this is not a k-ETF for $k = 1, 2$.

Table 2 Feasibility of the Kerdock set in generating a spanning set for $\mathrm{Sym}^k(\mathbb{C}^{2^\ell})$

k	$\dim(\mathrm{Sym}^k(\mathbb{C}^{2^\ell}))$	Set size (m)	Feasible?
1	2^ℓ	$2^\ell(2^\ell+1)$	Yes, for all ℓ.
2	$\frac{2^\ell(2^\ell+1)}{2}$	$2^\ell(2^\ell+1)$	Yes, for all ℓ.
3	$\frac{2^\ell(2^\ell+1)(2^\ell+2)}{6}$	$2^\ell(2^\ell+1)$	Only for $\ell \le 2$.

Proof We know that for any two matrices P and Q in $DG(\ell,0)$, $\mathrm{rank}(P-Q) = \ell$. The inner product given in (7) then implies that any two vectors coming from columns of distinct U_{P_i}s in Φ_{Ker} have inner product equal to $\frac{1}{\sqrt{2^\ell}}$ in modulus. Due to orthonormality, the modulus of the inner product between two different columns of the same matrix U_{P_i} or I_{2^ℓ} is zero. From (5), it follows that the modulus of the inner product between any column of I_{2^m} and a column of an U_{P_i} is $\frac{1}{\sqrt{2^\ell}}$.

Recall that the columns of Φ_{Ker} are vectors in $\mathbb{C}^{2^\ell}$, and that the columns of each U_{P_i} as well as I_{2^ℓ} form an orthonormal basis of $\mathbb{C}^{2^\ell}$. The inner product property thus shows that the set of columns of Φ_{Ker} is the union of $2^\ell+1$ mutually unbiased bases in $\mathbb{C}^{2^\ell}$. From [2, Theorem 3], the columns of Φ_{Ker} form a complex projective 2-design. By Theorem 1, this is a WBE set for $k = 1, 2$, and $\Phi_{\mathrm{Ker}}^{\otimes 2}$ is a tight frame for $\mathrm{Sym}^2(\mathbb{C}^{2^\ell})$. Since the set is not equiangular, it is not a k-ETF for $k = 1, 2$. □

The proof of Theorem 4 gives rise to the following.

Corollary 5 *The bases* $\{U_{P_{i_1}}, U_{P_{i_2}}, \cdots, U_{P_{i_{2^\ell}}}, I_{2^\ell}\}$ *form a maximal set of* $2^\ell+1$ *mutually unbiased bases in* $\mathbb{C}^{2^\ell}$.

It is noteworthy that the construction of maximal MUBs in $\mathbb{C}^{2^\ell}$ from Kerdock sets is mentioned in [24] (see Proposition 2.8 therein).

Even though the Kerdock set gives 2-designs and hence WBE sets for $k = 1, 2$, it is not feasible to extend these for values of k higher than 2 due to insufficient vectors; the number of vectors falls below $d(n,k)$ when $\ell > 2$ for $k > 2$. This is shown in Table 2. If we increase the set size beyond the Kerdock set by considering, say, $DG(m,1)$, then one might hope that the corresponding Φ in Theorem 4 will yield WBE sets for $k > 2$. Unfortunately, this is not the case; see Example 3 below.

Example 3 Let $\ell = 3$. The set $DG(3,1)$ has $2^{\ell(r+1)} = 2^{3\cdot(1+1)} = 64$ matrices, and hence accounts for all binary symmetric 3×3 matrices. Thus $DG(3,1)$ corresponds to the second order Reed-Muller code of size 2^3. The columns of the matrix

$$\Phi_1 = \begin{bmatrix} U_{P_1} & \cdots & U_{P_{64}} \end{bmatrix}$$

form a tight frame of $2^3 \times 64 = 512$ vectors in $\mathbb{C}^8$, and hence a WBE set for $k = 1$. However, they do not form a WBE set for $k = 2$ or for $k = 3$. This can be verified by using Matlab to compute the sum on the left side of (1). The same is true when the identity matrix is concatenated to Φ_1, i.e., the columns of the matrix

$$\Phi_2 = \begin{bmatrix} U_{P_1} & \cdots & U_{P_{64}} & I_8 \end{bmatrix}$$

form a WBE set of $512+8=520$ vectors in $\mathbb{C}^8$ for $k=1$, but they do not form a WBE set for $k=2$ or for $k=3$.

Example 4 An ubiquitous tool in quantum information theory is the stabilizer formalism, which was originally invented to describe quantum error correcting codes. Stabilizer states are joint eigenvectors of generalized Pauli matrices [25]. It has been shown in [25, Corollary 1] that the set of all ℓ-qubit (quantum bit) stabilizer states forms a complex projective 3-design in dimension 2^ℓ for all $\ell \in \mathbb{N}$. By Theorem 1, this therefore provides a source of WBE sets in $\mathbb{C}^{2^\ell}$ for $k=1,2,3$.

As mentioned above and shown in Table 2, for a given ℓ, the Kerdock set will not generate enough vectors to span $\mathrm{Sym}^k(\mathbb{C}^{2^\ell})$ when $k>2$. Example 3 shows that increasing the number of vectors by considering $DG(\ell,h)$, $1 \le h \le \frac{\ell-1}{2}$, may generate enough vectors to span $\mathrm{Sym}^k(\mathbb{C}^{2^\ell})$, however, the resulting set may not be a WBE set for $k>2$. For a WBE set of m vectors in $\mathbb{C}^n$, one has

$$\frac{\sum_{i=1}^m \sum_{j=1}^m |\langle x_i, x_j\rangle|^{2k}}{\frac{m^2}{\binom{n+k-1}{k}}} = 1. \tag{8}$$

When considering $DG(\ell,h)$ with $h \ge 1$, if equality is not attained in (1), it seems natural to ask how close one can get to attaining equality, i.e., how much the ratio in the left side of (8) diverges from 1. For $h=1$, an upper bound for this ratio can be calculated and is given below in Proposition 6.

Proposition 6 *Consider the set $DG(\ell,1)$, and let U_P denote the resulting unitary matrix coming from a P in $DG(\ell,1)$. Analogous to (6), denote by Φ the concatenation of the matrices U_P. Then Φ can be written as*

$$\Phi = \begin{bmatrix} U_{P_1} & U_{P_2} & \cdots & U_{P_{2^\ell}} & U_{P_{2^\ell+1}} & \cdots & U_{P_{2^{2\ell}}}\end{bmatrix}. \tag{9}$$

Denoting the jth column of Φ by x_j and the total number of columns by m,

$$1 \le \frac{\sum_{i=1}^m \sum_{j=1}^m |\langle x_i, x_j\rangle|^{2k}}{\frac{m^2}{\binom{2^\ell+k-1}{k}}} \le \frac{2^\ell(2^\ell+1)\cdots(2^\ell+k-1)}{k!2^{3\ell}}\left[1+\frac{2^\ell-1}{2^{\ell k}}+\frac{2^{3\ell}-2^{\ell+1}+1}{2^{\ell k}}2^{2k}\right]. \tag{10}$$

Proof Since $DG(\ell,h)$ has $2^{\ell(h+1)}$ matrices [21], $DG(\ell,1)$ has $2^{2\ell}$ matrices. Assume that the matrices $\{U_{P_i}\}_{i=1}^{2^{2\ell}}$ are arranged in such a way that the first 2^ℓ matrices P_i belong to the Kerdock set or $DG(\ell,0)$. The remaining matrices, i.e., $\{P_i\}_{i=2^\ell+1}^{2^{2\ell}}$ belong to $DG(\ell,1)\setminus DG(\ell,0)$. This gives Φ the structure in (9).

The $2^{2\ell}$ matrices P in $DG(\ell,1)$ give vectors in $\mathbb{C}^{2^\ell}$ by (5). Each unitary matrix U_{P_i} being $2^\ell \times 2^\ell$, this implies that the total number of columns in Φ is

$$m = 2^\ell 2^{2\ell} = 2^{3\ell}.$$

To obtain (10), we next look at the distinct possibilities of inner products coming from the columns of Φ.

(i) By construction, each column of Φ has unit norm. Thus the total number of inner products that equal one is $m = 2^{3\ell}$.

(ii) The columns of $\{U_{P_1}, \ldots, U_{P_{2^\ell}}\}$ form a set of 2^ℓ mutually unbiased bases; see Corollary 5. The inner product of columns coming from different bases is $\frac{1}{\sqrt{2^\ell}}$. The total number of inner products involving vectors of different bases is

$$(2^\ell - 1)2^\ell 2^\ell 2^\ell = (2^\ell - 1)2^{3\ell}.$$

(iii) The inner product of columns within the same basis is zero. There are $2^{2\ell}$ bases. For each basis there are $(2^\ell)^2 - 2^\ell$ inner products that are zero.

(iv) Due to (7), the inner products not accounted for in (i) - (iii) will be bounded above by $\frac{1}{\sqrt{2^{\ell-2}}}$.
The total number of inner products not accounted for in (i) - (iii) is

$$m^2 - \left[m + (2^\ell - 1)2^{3\ell} + 2^{3\ell}(2^\ell - 1)\right] = 2^{6\ell} - 2 \cdot 2^{4\ell} + 2^{3\ell}.$$

From (i) - (iv), we have

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{m} |\langle x_i, x_j\rangle|^{2k} &\le m + 2^{3\ell}(2^\ell - 1)\left(\frac{1}{\sqrt{2^\ell}}\right)^{2k} + (2^{6\ell} - 2\cdot 2^{4\ell} + 2^{3\ell})\left(\frac{1}{\sqrt{2^{\ell-2}}}\right)^{2k} \\ &= 2^{3\ell}\left[1 + (2^\ell - 1)2^{-\ell k} + (2^{3\ell} - 2^{\ell+1} + 1)2^{-k(\ell-2)}\right]. \end{aligned} \tag{11}$$

The Welch bound from (1) is

$$\frac{m^2}{\binom{n+k-1}{k}} = \frac{2^{6\ell}}{\binom{2^\ell+k-1}{k}}. \tag{12}$$

Taking the ratio of the right sides of (11) and (12) gives (10) after some simplification.

□

4 Frame properties of sets constructed from second order Reed-Muller codes

As mentioned in Sections 1 and 2, to get a k-ETF one needs an ETF, and these are known to be rare [13]. In an ETF, the inner product between any two distinct vectors takes only one value in modulus. In this section, the number of distinct inner products (in moduli) between vectors in sets obtained from the second order Reed-Muller code is investigated, see Theorem 8. In the process, some frame properties of the vectors that form the second order Reed-Muller code are discussed. The reader is referred to [26] for details on frame theory.

Lemma 7 *Consider the matrix* $\Phi = \left[U_{P_1}\ U_{P_2}\ \ldots\ U_{P_{2^{\ell(\ell+1)/2}}}\right]$ *introduced in (6).*
(i) The columns of Φ *form a unit norm tight frame for* $\mathbb{C}^{2^\ell}$ *with frame bound* $2^{\ell(\ell+1)/2}$.

(ii) Let N be an integer such that $1 \leq N \leq 2^{\ell(\ell+1)/2}$. Then $\cup_{n=1}^{N} U_{P_{i_n}}$, where $i_n \in \{1, 2, \ldots, 2^{\ell(\ell+1)/2}\}$, is a unit norm tight frame of $N2^{\ell}$ vectors in $\mathbb{C}^{2^{\ell}}$ with bound N.

Proof (i) Each matrix U_{P_i} appearing in Φ is an orthonormal basis of $\mathbb{C}^{2^{\ell}}$. Recall that an orthonormal basis is a tight frame with frame bound equal to one. Also, if Φ_1 and Φ_2 are two tight frames with frame bounds A_1 and A_2, respectively, then their union is also a tight frame with frame bound equal to $A_1 + A_2$. The columns of Φ is the union of $2^{\ell(\ell+1)/2}$ orthonormal bases and hence form a unit norm tight frame with bound equal to $2^{\ell(\ell+1)/2}$.
The proof of (ii) is identical to that of (i) by considering a subset of the U_P matrices. □

The tightness of the frames given in Lemma 7 is not preserved when extended to $\mathrm{Sym}^k(\mathbb{C}^{2^{\ell}})$, by taking tensor powers of the columns. This is exhibited in Example 3 by keeping in mind the equivalence of tight frames and WBE sets (see Theorem 1). The columns of Φ are not equiangular and hence these do not give a k-ETF ($k \geq 1$) even though they form a WBE set; see the text immediately before Theorem 1 and Example 2, as well as Lemma 3. By considering vectors associated with the $DG(\ell, h)$ sets (see Section 3), one can restrict the number of distinct inner products among vectors by considering a lower value of h. This is shown in Theorem 8 below.

Theorem 8 *Let ℓ be an odd positive integer.*
(i) For $P_{i_k} \in DG(\ell, 0)$, $1 \leq k \leq 2^{\ell}$, consider

$$\Phi_{Ker} = \left[U_{P_{i_1}} \; U_{P_{i_2}} \; \cdots \; U_{P_{i_{2^\ell}}} \right].$$

The columns of Φ_{Ker} form a unit norm tight frame with two distinct inner product in moduli among distinct vectors.
(ii) If the domain of P is $DG(\ell, h)$ then the columns of the matrix obtained by concatenating the unitary matrices $\{U_P\}_{P \in DG(\ell,h)}$ form a unit norm tight frame with $2h + 2$ distinct inner products in modulus, where $0 \leq h \leq \frac{\ell-1}{2}$.
(iii) The unit norm tight frame in (i) of Lemma 7 has $\ell + 1$ distinct inner products in modulus.

Proof From Lemma 7, it follows that the columns form a unit norm tight frame in each case. We just have to count the number of distinct inner products, and this can be done using (7).
(i) When the P matrices come from $DG(\ell, 0)$ then by (7), the inner product between distinct vectors is either 0 or $\frac{1}{\sqrt{2^{\ell}}}$.
(ii) Let the domain of P be $DG(\ell, h)$, $0 \leq h \leq \frac{\ell-1}{2}$. Recall that the rank of the difference between any two matrices in $DG(\ell, h)$ is at least $\ell - 2h$. Then, by (7), the set of possible inner product values in modulus for distinct vectors is $\{0, 2^{-\ell/2}, 2^{-(\ell-1)/2}, \ldots, 2^{-(\ell-2h)/2}\}$. Thus there are $2h + 2$ distinct inner products in modulus.

(iii) For the tight frame in (i) of Lemma 7, the domain of P is the entire DG set, i.e., $DG(\ell, \frac{\ell-1}{2})$. By setting $h = \frac{\ell-1}{2}$ in (ii) of this proof, we conclude that there are $\ell + 1$ distinct inner products in modulus. □

Remark 3 As discussed in [27], one can expect reasonably low coherence from tight frames if the number of distinct inner products in modulus can be restricted, provided that each inner product modulus arises the same number of times. In the absence of ETFs, unit norm tight frames with restricted number of inner products can be used to optimize the coherence among vectors in a frame.

Remark 4 In some applications, it is useful to use frames or vectors that are binary valued. In this regard, consider the vectors given by (5). If we only consider P matrices that have zero diagonal and ignore the normalization factor $\frac{1}{\sqrt{2^\ell}}$, then the components of the vectors $\phi_{P,b}$ in (5) are all ± 1. This allows us to get frames of binary vectors that have restricted number of angles. The codes obtained under these assumptions are the same as would be obtained by generating complex codes of length $2^{\ell-1}$ and applying the Gray map [20].

Acknowledgments. The author is grateful to Doug Cochran for many useful discussions on the topic of this article. The author would like to acknowledge support from the Simons Foundation under Award Number 709212. The author also wishes to thank the anonymous referees for their suggestions.

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