Tight frames in spiral sampling

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***Abstract*—The paper deals with the construction of Parseval frames for $L^2(B(0,R))$, the space of square integrable functions whose domain is the ball of radius R. The focus is on Fourier frames on a spiral. Starting with a Fourier frame on a spiral, a Parseval frame that spans the same space can then be obtained by a symmetric approximation of the original Fourier frame.**

I. Introduction

Earlier work by Benedetto et al. [1], [2], [3], [4] gave the construction of a set of points on a given spiral such that these points give rise to a frame for $L^2(B(0,R))$, the space of all square integrable functions on the ball centered at the origin and of radius R. This means that given a spiral A_c, the authors in [1], [2], [3], [4] were able to construct a sequence of points Λ on this spiral and its interleaves such that every signal f belonging to $L^2(B(0,R))$ can be written as $\sum_{\lambda\in\Lambda} a_\lambda(f)e_\lambda$ where $e_\lambda(x) = e^{2\pi i x\cdot\lambda}$. The incentive of choosing points on a spiral comes from the applicability in MRI (Magnetic Resonance Imaging) where a signal is sampled in the Fourier domain along interleaving spirals, resulting in fast imaging methods. For practical purposes, the reconstruction of signals using such infinite frames entails inverting the frame operator and/or using only finitely many samples. Such numerical issues are mitigated if one can use a tight frame. The possibility of expanding a function as a non-harmonic Fourier series was discovered by Paley and Wiener. For a sequence Λ of real numbers, it is natural to ask whether every band-limited signal with spectrum E can be reconstructed in a stable way from its samples $\{F(\lambda), \lambda \subseteq \Lambda\}$. Landau [5] proved a necessary condition for $\{e^{2\pi i x\cdot\lambda}, \lambda\in\Lambda\}$ to be a frame for the space of band-limited functions with spectrum E by relating the lower density of Λ to the measure of E. There is an extensive literature on the stable reconstruction problem, (see, e.g., [6], [7], [8], [9], [10], [11]). Many of the contributions to this area focus on the theoretical aspect, while our emphasis is on explicit construction.

The main contribution of this article is to give an explicit procedure to convert a frame which is not a tight frame into a Parseval frame, with the requirement that each element in the resulting Parseval frame can be expressed as a linear combination of the elements in the original frame. To be precise, this requirement means that if $\{f_1, f_2, f_3\}$ is the original frame for the Hilbert space $\mathcal{H}$, and $\{g_1, g_2, g_3\}$ is the resulting Parseval frame, then each g_n is a linear combination of f_1, f_2, f_3. For any function $f \in \mathcal{H}$, one has $f = \sum_{n=1}^{3}\langle f, g_n\rangle g_n$. Since each g_n is a linear combination of f_1, f_2, and f_3, each number $\langle f, g_n\rangle$ can be calculated from the three numbers $\langle f, f_1\rangle, \langle f, f_3\rangle, \langle f, f_3\rangle$. Hence, from the numbers $\langle f, f_n\rangle$ for $n = 1, 2, 3$, one can recover f. In the reconstruction formula using the Parseval frame, only the measurements obtained from the original frame are needed. This feature is extremely important, especially in the aforementioned application to MRI, when the measurements from the original frame are the only available measurements. The procedure explained in this article applies to other frames, and not just to Fourier frames, but motivated by applications to medical imaging as in MRI, the focus here is only on spiral sampling with Fourier frames.

In [12], Frank, Paulsen, and Tiballi obtain a Parseval frame from a given frame that spans the same subspace as the original frame and is closest to it in some sense, which they call *symmetric approximation*. The approach used in [12] is to use the polar decomposition of the synthesis operator of the original frame. This idea inspires the method developed in the present work to obtain Parseval frames for the spiral sampling case. Presently, the work is only focused on finite frames. The symmetric approximation of infinite Fourier frames on spirals and the best N-term approximation of such frames constitute ongoing research.

A. Notation and preliminaries

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, and let $\widehat{\mathbb{R}}^d$ denote $\mathbb{R}^d$ when it is considered as the domain of the Fourier transforms of signals defined on $\mathbb{R}^d$. $L^2(\widehat{\mathbb{R}}^d)$ is the space of square integrable functions ϕ on $\widehat{\mathbb{R}}^d$, i.e.,

$$||\phi||_{L^2(\widehat{\mathbb{R}}^d)} = \left(\int_{\widehat{\mathbb{R}}^d} |\phi(\gamma)|^2 \mathrm{d}\gamma\right)^{1/2} < \infty,$$

$\phi^\vee$ is the inverse Fourier transform of ϕ defined as

$$\phi^\vee(x) = \int_{\widehat{\mathbb{R}}^d} \phi(\gamma)e^{2\pi i x\cdot\gamma}\mathrm{d}\gamma,$$

and *supp* $\phi^\vee$ denotes the support of $\phi^\vee$. Let $E \subseteq \widehat{\mathbb{R}}^d$ be closed. The *Paley-Wiener* space PW_E is

$$PW_E = \{\phi \in L^2(\widehat{\mathbb{R}}^d) : supp\ \phi^\vee \subseteq E\}.$$

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{f_n : n \in \mathbb{Z}^d\} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq$

$B < \infty$ such that

$$\forall y \in \mathcal{H}, \quad A||y||^2 \le \sum_n |\langle y, f_n\rangle|^2 \le B||y||^2.$$

The constants A and B are called the lower and upper frame bounds, respectively. If $A = B$, the frame is said to be *tight* and if $A = B = 1$, the frame is called a *Parseval* frame. Let $\{f_n\}$ be a frame for $\mathcal{H}$. The *synthesis operator* is the linear mapping $T : \ell_2 \to \mathcal{H}$ given by $T(\{c_i\}) = \sum_k c_k f_k$. The frame operator $S : \mathcal{H} \to \mathcal{H}$ is TT^* and is given by

$$\forall y \in \mathcal{H}, \quad S(y) = \sum_n \langle y, f_n\rangle f_n.$$

For every $y \in \mathcal{H}$,

$$y = \sum_n \langle y, S^{-1} f_n\rangle f_n = \sum_n \langle y, f_n\rangle S^{-1} f_n.$$

For more on frames one can look at [13] or [14].

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a sequence and let $E \subset \mathbb{R}^d$ have finite Lebesgue measure. By the Parseval Formula, the following are equivalent ([3], [4]).

(i) $\{e_\lambda : \lambda \in \Lambda\}$ is a frame for $L^2(E)$.

(ii) There exist $0 < A \leqslant B < \infty$ such that

$$A||\phi||_2^2 \leqslant \sum_{\lambda\in\Lambda} |\phi(\lambda)|^2 \leqslant B||\phi||_2^2,$$

for all ϕ in PW_E. In this case, we say that Λ is a *Fourier frame* for PW_E.

A set Λ is *uniformly discrete* if there exists $r > 0$ such that

$$\forall \lambda, \gamma \in \Lambda, \quad |\lambda - \gamma| \ge r,$$

where $|\lambda - \gamma|$ is the Euclidean distance between λ and γ.

If for two frames $\{f_i\}_{i\in\mathbb{N}}$ and $\{g_i\}_{i\in\mathbb{N}}$ of two Hilbert subspaces $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{H}$, respectively, there exists an invertible bounded linear operator $T : \mathcal{K} \to \mathcal{L}$ such that $T(f_i) = g_i$ for every index i, then these two frames are said to be *weakly similar* [12]. A Parseval frame $\{\nu_i\}_{i=1}^n$ in a finite dimensional Hilbert subspace $\mathcal{L} \subseteq \mathcal{H}$ is said to be a *symmetric approximation* of a finite frame $\{f_i\}_{i=1}^n$ in a Hilbert subspace $\mathcal{K} \subseteq \mathcal{H}$ if the frames $\{f_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ are weakly similar and the inequality

$$\sum_{j=1}^n \|\mu_j - f_j\|^2 \ge \sum_{j=1}^n \|\nu_j - f_j\|^2$$

is valid for all Parseval frames $\{\mu_i\}_{i=1}^n$ in Hilbert subspaces of $\mathcal{H}$ that are weakly similar to $\{f_i\}_{i=1}^n$ [12]. If $\mathcal{K} = \mathcal{L}$, the frames are called *similar*.

When a 3 by 3 matrix W is acting on a sequence of elements $\{f_1, f_2, f_3\}$, this action is denoted by $\{e_1, e_2, e_3\} = W \cdot \{f_1, f_2, f_3\}$, or in matrix notation,

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{23} & w_{33} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

to denote

$$e_1 = \sum_{j=1}^3 w_{1j} f_j, \; e_2 = \sum_{j=1}^3 w_{2j} f_j, \; e_3 = \sum_{j=1}^3 w_{3j} f_j.$$

B. Background

The following theorem [1], [2], [3], [4] is based on a deep result of Beurling [15].

Theorem I.1 (Beurling Covering Theorem). *Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be uniformly discrete, and define $\rho = \sup_{\mu\in\widehat{\mathbb{R}}^d} \operatorname{dist}(\mu, \Lambda)$ where $\operatorname{dist}(\mu, \Lambda)$ is the Euclidean distance between the point μ and the set Λ. If $R\rho < 1/4$, then Λ is a Fourier frame for $\mathrm{PW}_{B(0,R)}$.*

In [1], [2], [3], [4] the authors have used the Beurling Covering Theorem to give an explicit construction of Fourier frames from points that lie on a spiral. In particular, the following result can be found in [2].

Example I.2. *Fix $c > 0$. In $\widehat{\mathbb{R}}^2$, consider the spiral*

$$A_c = \{c\theta \cos 2\pi\theta, c\theta \sin 2\pi\theta : \theta \ge 0\}.$$

For R and δ satisfying $Rc < 1/2$ and $(\frac{c}{2} + \delta)R < 1/4$, one chooses a uniformly discrete set of points Λ such that the curve distance between any two consecutive points is less than 2δ, and beginning within 2δ of the origin. Then Λ satisfies the Beurling Covering Theorem and hence gives rise to a Fourier frame for $\mathrm{PW}_{B(0,R)}$.

The synthesis operator T defined in Section I-A is bounded and has a natural polar decomposition $T = W|T|$, where W is a partial isometry from ℓ_2 into $\mathcal{H}$. To obtain a symmetric approximation of a given frame, the following has been shown in [12].

Theorem I.3. *Let $\{\mu_i\}_{i=1}^n$ be a Parseval frame in a Hilbert subspace $\mathcal{L} \subseteq \mathcal{H}$ and let $\{f_i\}_{i=1}^n$ be a frame in a Hilbert subspace $\mathcal{K} \subseteq \mathcal{H}$ such that both these frames are weakly similar. Letting the standard orthonormal basis for $\mathbb{C}^n$ be denoted by $\{e_i\}_{i=1}^n$, the following inequality*

$$\sum_{j=1}^n \|\mu_j - f_j\|^2 \ge \sum_{j=1}^n \|W(e_j) - f_j\|^2$$

holds. Equality appears if and only if $\mu_j = W(e_j)$ for $j = 1, \ldots, n$. (Consequently, the symmetric approximation of a frame $\{f_i\}_{i=1}^n$ in a finite dimensional Hilbert space $\mathcal{K} \subseteq \mathcal{H}$ is a Parseval frame spanning the same Hilbert subspace $\mathcal{L} \equiv \mathcal{K}$ of $\mathcal{H}$ and being similar to $\{f_i\}_{i=1}^n$.)

Similar results for infinite frames in separable Hilbert spaces have also been established in [12] but for now the focus is on the finite dimensional case.

II. Parseval frames from a finite Fourier frame on a spiral

Three examples are discussed below. In the first two examples, the frame under consideration is on $\widehat{\mathbb{R}}$. The third example is for a Fourier frame on a spiral in $\widehat{\mathbb{R}}^2$.

In the first two examples, the procedure suggested by Theorem I.3 is modified so that in the final step, matrix multiplication is replaced by a matrix acting on a sequence of elements in a Hilbert space.

Example II.1. *Let* $\{f_1 = e^{2\pi i\lambda_1 x}, f_2 = e^{2\pi i\lambda_2 x}, f_3 = e^{2\pi i\lambda_3 x}\}$ *be a frame that spans a subspace of* $L^2([-1/2, 1/2])$. *Choose* $\lambda_1 = 3 + \frac{1}{3}, \lambda_2 = 4 + \frac{1}{4}, \lambda_3 = 5 + \frac{1}{5}$.

This frame is used to construct a Parseval frame that spans the same subspace. Let $\mathcal{H}$ *be the span of* $\{f_1, f_2, f_3\}$ *and let* $\{e_1, e_2, e_3\}$ *be an orthonormal basis of* $\mathcal{H}$. *One can construct an orthonormal basis* $\{e_1, e_2, e_3\}$ *by applying the Gram-Schmidt orthogonalization process to* $\{f_1, f_2, f_3\}$. *The resulting orthonormal basis can be written as*

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -c_{21} & 1 & 0 \\ c_{21}\theta - c_{31} & -\theta & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

where

$$c_{21} = sinc(\lambda_2 - \lambda_1), c_{32} = sinc(\lambda_3 - \lambda_2), c_{31} = sinc(\lambda_3 - \lambda_1),$$

and

$$sinc(x) \equiv \frac{\sin(\pi x)}{\pi x}, \ \theta = \frac{c_{32} - c_{21}c_{31}}{1 - c_{21}^2}.$$

Then

$$\begin{aligned} f_1 &= e_1, \\ f_2 &= c_{21}e_1 + e_2, \\ f_3 &= c_{31}e_1 + \theta e_2 + e_3, \end{aligned}$$

and the synthesis operator T *of the frame* $\{f_1, f_2, f_3\}$ *can be written in matrix form as*

$$\begin{bmatrix} 1 & c_{21} & c_{31} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Next the polar decomposition of the matrix of T *is computed, so that* $T = W|T|$, *where* W *is a partial isometry and* $|T| = (T^*T)^{1/2}$. *In this case, since* T *is invertible,* W *is in fact a unitary matrix. Finally, let* $\{g_1, g_2, g_3\} = W^* \cdot \{e_1, e_2, e_3\}$. *Then* $\{g_1, g_2, g_3\}$ *forms a Parseval frame for* $\mathcal{H}$.

Remark: (1). In this example, since the original frame is linearly independent and therefore a basis for $\mathcal{H}$, *what is obtained as a Parseval frame is in fact an orthonormal basis for* $\mathcal{H}$. *(2). Since each* g_n *can be written as a linear combination of* f_1, f_2, *and* f_3, *the Parseval frame constructed indeed spans the same subspace as the original frame.*

Example II.2. *Let* $\lambda_1 = 3 + \frac{1}{3}, \lambda_2 = 4 + \frac{1}{4}, \lambda_3 = 5 + \frac{1}{5}$ *and let* $f_1 = e^{2\pi i\lambda_1 x}, f_2 = e^{2\pi i\lambda_2 x}, f_3 = e^{2\pi i\lambda_3 x}, f_4 = f_1 + f_2, f_5 = f_1 + f_3$, *and* $f_6 = f_2 + f_3$. *Consider the frame* $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ *of a subspace of* $L^2([-1/2, 1/2])$. *Denote this subspace by* $\mathcal{H}$. *Starting from the linearly independent set* $\{f_1, f_2, f_3\}$ *that spans* $\mathcal{H}$, *one can construct an orthonormal basis* $\{e_1, e_2, e_3\}$ *for* $\mathcal{H}$ *as done in Example II.1. From Example II.1,*

$$\begin{aligned} f_1 &= e_1, \\ f_2 &= c_{21}e_1 + e_2, \\ f_3 &= c_{31}e_1 + \theta e_2 + e_3, \\ f_4 &= f_1 + f_2 = (1 + c_{21})e_1 + e_2, \\ f_5 &= f_1 + f_3 = (1 + c_{31})e_1 + \theta e_2 + e_3, \\ f_6 &= f_2 + f_3 = (c_{21} + c_{31})e_1 + (1 + \theta)e_2 + e_3, \end{aligned}$$

where c_{21}, c_{31}, *and* θ *are as defined in Example II.1. The synthesis operator* T *has the matrix representation*

$$\begin{bmatrix} 1 & c_{21} & c_{31} & 1 + c_{21} & 1 + c_{31} & c_{21} + c_{31} \\ 0 & 1 & \theta & 1 & \theta & 1 + \theta \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Let the polar decomposition of T *be given by* $T = W|T|$. *Let* $\{g_1, g_2, g_3, g_4, g_5, g_6\} = W^* \cdot \{e_1, e_2, e_3\}$. *Note that* W^* *is a 6 by 3 matrix. Then it can be shown that* $\{g_k\colon 1 \le k \le 6\}$ *forms a Parseval frame for* $\mathcal{H}$.

Example II.3. *A Fourier frame of three elements is first constructed using Example I.2. Let* $c = 1$, $R = 1/4$, *and* $\delta = 1/4$. *Three points on the spiral* $A_{c=1} = \{\theta\cos 2\pi\theta, \theta\sin 2\pi\theta\}$ *that have arc-length between them less than* 2δ, *starting with* 2δ *from the origin, can be obtained by taking three values of* θ *to be* $\theta_1 = 1/16$, $\theta_2 = 1/8$, *and* $\theta_3 = 1/4$. *This choice gives the following three points on the spiral*

$$\lambda_1 = (\frac{1}{16}\cos\frac{\pi}{8}, \frac{1}{16}\sin\frac{\pi}{8}) = (0.06, 0.02),$$

$$\lambda_2 = (\frac{1}{8}\cos\frac{\pi}{4}, \frac{1}{8}\sin\frac{\pi}{4}) = (0.09, 0.09),$$

and

$$\lambda_3 = (\frac{1}{4}\cos\frac{\pi}{2}, \frac{1}{4}\sin\frac{\pi}{2}) = (0, 1/4).$$

Thus $X = \{e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3}\}$ *is a Fourier frame for* $\mathrm{span}\{e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3}\}$.

For implementation purposes, to get the symmetric approximation, one can think of discretizing the ball $B(0, 1/4)$ *by changing into polar coordinates and looking at the rectangle* $\{(r, \theta) : 0 \le r \le 1/4, 0 \le \theta \le 2\pi\}$. *One can then divide each side of the rectangle into* N *subintervals partitioning it into* N^2 *rectangles. The exponential functions from the set* X *are then evaluated at* N^2 *grid-points, taking one point from each small rectangle and thus obtaining a vector* v_i *of length* N^2 *for each* e_{λ_i}, $i = 1, 2, 3$. *Looking at the synthesis operator* F *of* X *as the matrix* $[F]$ *whose columns are* v_i; *such a matrix will be of size* N^2 *by 3. After carrying out the polar decomposition of* $[F]$ *using Matlab, one can get the discretized Parseval frame* $\{u_i\}_{i=1}^3$ *that will be considered as the symmetric approximation of the above Fourier frame.*

Suppose one is interested in reconstructing a function f *in* $\mathrm{span}\{e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3}\}$. *First* f *is converted into a vector* $[f]$ *of size* N^2 *by evaluating it at the* N^2 *points on the rectangular grid above. Then* f *is reconstructed at the* N^2 *points as*

$$[\tilde{f}] = \sum_{j=1}^{3} \langle [f], u_i \rangle u_i.$$

The results are shown in Figures 1 and 2 for the reconstruction of $f = e_{\lambda_1}$ and $f = e_{\lambda_1} - 2e_{\lambda_2} + e_{\lambda_3}$, respectively. Only the real part of the original and the reconstructed functions are plotted. Also, for clarity of reading the figures, only a certain number of points are plotted instead of all the N^2 points.

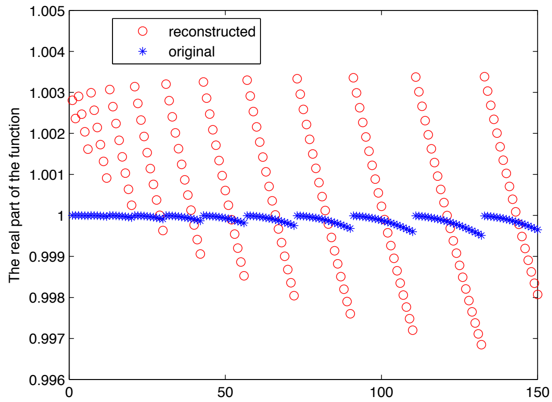

Fig. 1. Reconstruction of the function $f = e_{\lambda_1}$ using $N = 50$.

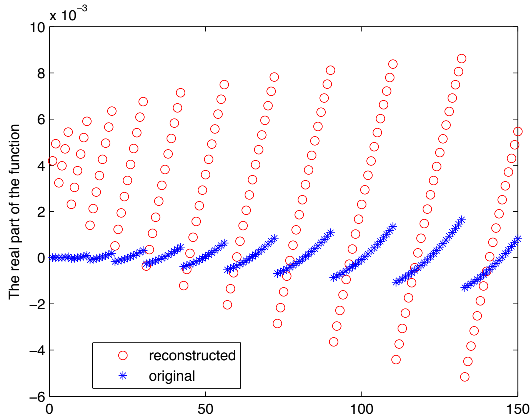

Fig. 2. Reconstruction of the function $f = e_{\lambda_1} - 2e_{\lambda_2} + e_{\lambda_3}$ using $N = 50$.

III. Conclusion

In this paper, the construction of a Parseval frame that is a symmetric approximation of a Fourier frame on a spiral has been considered. Presently, the focus is only on finite frames. This is done by means of the polar decomposition of the matrix corresponding to the synthesis operator of the Fourier frame. The reconstruction of functions lying in the span of such Fourier frames on spirals has been studied. By using a Parseval frame that spans the same space as the original Fourier frame, the reconstruction avoids the need to compute the inverse of the frame operator of the original frame. Besides, the Parseval frame that is obtained by considering the symmetric approximation enables one to reconstruct a function by only using the measurements obtained from the original Fourier frame.

Finding a Parseval frame for some general separable Hilbert space that is a symmetric approximation of a given frame involves finding the polar decomposition of the synthesis operator. This constitutes ongoing research. For practical purposes, even after finding a Parseval frame, it is not possible to use an infinite frame and one should think of finding the best N-term approximation. This will be a part of future research.

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