Adv. Pure Appl. Math. **3** (2012), 421–441
DOI 10.1515/apam-2012-0028 

# Sampling of homogeneous polynomials and approximating multivariate functions

Somantika Datta, Stephen Howard and Douglas Cochran

**Abstract.** Conditions for reconstruction of multivariate homogeneous polynomials from sets of sample values are introduced. In addition, it is shown that one can explicitly obtain the polynomial coefficients from the sample data by considering frames for spaces of symmetric tensors. Further, it is discussed how the reconstruction of homogeneous polynomials can be used to approximate certain smooth functions.

**Keywords.** Frames, homogeneous polynomials, sampling, symmetric tensors.

**2010 Mathematics Subject Classification.** 94A20, 42C15.

## 1 Introduction

### 1.1 Motivation

Several authors have noted the importance of interpolation and reconstruction of multivariate polynomials from sample data in applications. Zakhor [13], for example, considered the problem of interpolation of bivariate polynomials from irregularly spaced sample values in connection with two-dimensional filter design and image processing. The case of multivariate polynomials presents significant difficulties not encountered with polynomials of one variable, in particular due to the zeros of these entire functions of several variables not being isolated as occurs in the univariate setting. Consequently, it is not surprising that, in her work, Zakhor [13] develops conditions in which suitable sampling sets are constrained to lie on certain algebraic curves.

More recent work by Varjú [12] and Benko and Króo [1] develops Weierstraß types of results for approximation of smooth multivariate functions by homogeneous polynomials. This suggests the potential utility of interpolation and reconstruction of homogeneous polynomials from sample values. It is well known that the linear space $H_k(\mathbb{C}^n)$ of homogeneous polynomials of degree $k$ in $n$ complex variables is isomorphic to the space $\mathrm{Sym}^k(\mathbb{C}^n)$ of symmetric $k$-tensors over $\mathbb{C}^n$. This fact was used by the authors in [4] to develop results concerning frames and Grammians on $\mathrm{Sym}^k(\mathbb{C}^n)$. In this paper, a similar perspective is used to derive conditions under which coefficients of a multivariate homogeneous polynomial of *known degree* can be reconstructed explicitly from sets of sample values. Further, it is noted that, modulo general position issues, the number of samples is the crucial issue in determining suitability of a sampling set. Nevertheless, some sampling sets are "better" than others in that they provide snugger frames and hence the numerical advantages they entail. The relative merits of sampling sets in this respect do not depend on the particular polynomial to be reconstructed, thus allowing generically good sampling sets to be designed before any sampling is actually carried out.

## 1.2 Outline

The paper is divided as follows. In Section 2 it is shown that one can reconstruct the coefficients of a degree-$k$ homogeneous polynomial from its samples at points which form a frame for the space $\mathrm{Sym}^k(\mathbb{C}^n)$. This is stated in Theorem 2.1. Theorem 2.3 of Section 2 shows that a sampling set that suffices for $n$-variate homogenous polynomials of degree $k$ is also suitable for reconstructing the coefficients of any homogeneous polynomial in $n$ variables of degree $1 \leqslant \ell < k$. The converse is "almost always" true provided that the size of the sampling set is big enough. Section 3 gives several examples to demonstrate these ideas. Section 4 proposes a way of using this reconstruction of homogeneous polynomials to approximate certain smooth functions from given samples.

## 1.3 Notation

Before beginning the mathematical sections of the paper, a few comments on notation and terminology are needed. For two vectors $x = [x^{(1)}, \dots, x^{(n)}]^\top$ and $y = [y^{(1)}, \dots, y^{(n)}]^\top$ in $\mathbb{C}^n$, their inner product will be denoted by

$$\langle x, y \rangle = \sum_{j=1}^{n} \overline{x^{(j)}} y^{(j)}$$

where the bar denotes complex conjugate, i.e., the inner product is conjugate linear in its first argument and linear in its second argument. The corresponding convention will be used for inner products in other complex Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and let $X = \{x_k : k \in \mathcal{K}\}$, where $\mathcal{K}$ is some index set, be a collection of vectors in $\mathcal{H}$. Then $X$ is said to be a *frame* for $\mathcal{H}$ if there exist constants $A$ and $B$, $0 < A \leq B < \infty$, such that for any $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{k \in \mathcal{K}} |\langle x_k, f \rangle|^2 \leq B\|f\|^2.$$

The constants $A$ and $B$ are called the *frame bounds*, $A$ being the lower frame bound and $B$ the upper frame bound. If $A = B$, the frame is called a *tight frame*. For a finite dimensional Hilbert space, a finite spanning set is equivalent to a frame. Given a finite frame $X = \{x_1, \dots, x_m\}$ for an $n$-dimensional complex vector space $V$, the function $F : V \to \ell_2(\{1, \dots, m\}) = \mathbb{C}^m$ given by

$$F(w) = [\langle x_1, w\rangle, \dots, \langle x_m, w\rangle]^{\mathsf{T}}$$

will be called the *Bessel map* associated with $X$, while $\mathcal{F} = F^*F : V \to V$ (i.e., the composition of the adjoint of $F$ with $F$) will be called the *frame operator* associated with $X$. The *Gram matrix* or the *Grammian* of $X$ is $G_X = FF^*$. The Grammian and the frame operator have the same non-zero eigenvalues. The minimum and the maximum eigenvalues give the optimal lower and upper frame bounds, respectively [3]. Every $v \in V$ can be represented as [3]

$$v = \sum_{i=1}^{m} \langle \mathcal{F}^{-1} x_i, v\rangle x_i = \sum_{i=1}^{m} \langle x_i, v\rangle \mathcal{F}^{-1} x_i. \tag{1.1}$$

The sequence $\{\mathcal{F}^{-1} x_i\}_{i=1}^{m}$ is also a frame and is called the *canonical dual* of $X$. Besides, if $A$ and $B$ are the optimal lower and upper frame bounds for $X$, then the operator norm $\|\mathcal{F}^{-1}\|$ is bounded above by $\frac{1}{A}$.

The $k$-fold tensor product $V^{\otimes k}$ of an $n$-dimensional vector space $V$ is a vector space spanned by elements of the form $v_1 \otimes \cdots \otimes v_k$ where each $v_i \in V$ (see [6, 11]). The vector $v_1 \otimes \cdots \otimes v_k$ is called a tensor and has $n^k$ coordinates

$$(v_1^{(\ell_1)}, v_2^{(\ell_2)}, \dots, v_k^{(\ell_k)}),$$

where $\ell_i = 1, 2, \dots, n$, $i = 1, 2, \dots, k$, and $v_i^{(\ell)}$ denotes the $\ell^{\text{th}}$ coordinate of the vector $v_i$. A choice of basis $\{e_1, \dots, e_n\}$ for $V$ gives rise to a basis for $V^{\otimes k}$ consisting of the $n^k$ product elements $e_{i_1 \dots i_k} \equiv e_{i_1} \otimes \cdots \otimes e_{i_k}$, $1 \le i_1, \dots, i_k \le n$. In particular, $V^{\otimes k}$ has dimension $n^k$.

The space of symmetric $k$-tensors associated with $V$, denoted by $\mathrm{Sym}^k(V)$, is the subspace of $V^{\otimes k}$ consisting of those tensors which remain fixed under permutation. Specifically, denote by $S_k$ the symmetric group on $k$ symbols and define an action of $S_k$ on $V^{\otimes k}$ by

$$A_\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

Then $\mathrm{Sym}^k(V)$ consists of all elements of $V^{\otimes k}$ such that

$$A_\sigma(v_1 \otimes \cdots \otimes v_k) = v_1 \otimes \cdots \otimes v_k \quad \text{for all } \sigma \in S_k$$

(see [11, Chapter 10]). The symmetrizer of a tensor is an average of all the permutations. For $\{v_1, \dots, v_k\}$, the symmetrizer of $v_1 \otimes \cdots \otimes v_k$ is given by

$$\frac{1}{k!} \sum_{\sigma \in S_k} A_\sigma (v_1 \otimes \cdots \otimes v_k). \tag{1.2}$$

The symmetrizer gives the same symmetric tensor for all tensors involving the same set of vectors. The space $\mathrm{Sym}^k(V)$ is spanned by the tensor powers $v^{\otimes k}$ where $v \in V$. If $V$ has dimension $n$, then

$$\dim \mathrm{Sym}^k(V) = \binom{n+k-1}{k}.$$

One should note that the tensor $v_1 \otimes \cdots \otimes v_k$ and its symmetrizer created by (1.2) both lie in $V^{\otimes k}$ that has dimension

$$n^k > \binom{n+k-1}{k}.$$

One can thus identify the like coordinates in the symmetrizer to get a vector with $\binom{n+k-1}{k}$ coordinates. This is denoted by $v_1 \odot \cdots \odot v_k$. For example, if

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$$

are two vectors in $\mathbb{C}^2$, then the symmetrizer is given by

$$\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) = \begin{pmatrix} ac \\ \frac{1}{2}(bc + ad) \\ \frac{1}{2}(bc + ad) \\ bd \end{pmatrix}$$

which is a vector in $\mathbb{C}^{2^{\otimes 2}}$. Identifying the second and the third component of this symmetrizer gives the vector

$$v_1 \odot v_2 = \begin{pmatrix} ac \\ bc + ad \\ bd \end{pmatrix}$$

and this has three coordinates which is the same as the dimension of $\mathrm{Sym}^2(\mathbb{C}^2)$. $\mathrm{Sym}^k(V)$ has a natural inner product with the property

$$\left\langle v^{\otimes k}, w^{\otimes k} \right\rangle_{\mathrm{Sym}^k(V)} = \langle v, w \rangle_V^k . \tag{1.3}$$

## 2 Sampling of homogeneous polynomials

It is well known (see, e.g., [11]) that $H_k(\mathbb{C}^n)$, the linear space of homogeneous polynomials of total degree $k$ in variables $\bar{z}^{(1)}, \dots, \bar{z}^{(n)}$ is isomorphic to $\mathrm{Sym}^k(V)$. This section points out a connection between the condition that

$$X^{(k)} = \{x_1^{\otimes k}, \dots, x_m^{\otimes k}\}$$

is a frame for $\mathrm{Sym}^k(V)$ and the reconstructability of polynomials in $H_k(\mathbb{C}^n)$ from the values they take at sets of $m$ points in $\mathbb{C}^n$.

Beginning with $k = 1$, let $w \in V = \mathrm{Sym}^1(V)$ and denote by

$$[w^{(1)}, \dots, w^{(n)}]^\top \in \mathbb{C}^n$$

the coordinates of $w$ in some orthonormal basis for $V$. There is an obvious isomorphism that takes $w \in V$ to the polynomial $p_w \in H_1(\mathbb{C}^n)$ defined by

$$p_w(z^{(1)}, \dots, z^{(n)}) = w^{(1)}\bar{z}^{(1)} + \cdots + w^{(n)}\bar{z}^{(n)}.$$

If $X = \{x_1, \dots, x_m\}$ is a frame for $V$, the associated Bessel map $F : V \to \mathbb{C}^m$ is given by

$$F(w) = \begin{bmatrix} \langle x_1, w\rangle \\ \vdots \\ \langle x_m, w\rangle \end{bmatrix} = \begin{bmatrix} p_w(x_1^{(1)}, \dots, x_1^{(n)}) \\ \vdots \\ p_w(x_m^{(1)}, \dots, x_m^{(n)}) \end{bmatrix}. \tag{2.1}$$

In other words, $F(w)$ is a vector of values obtained by evaluating (i.e., "sampling") $p_w$ at the points $x_1, \dots, x_m$. One may ask whether this set of $m$ sample values is sufficient to uniquely determine $p_w$.

To address this question, define a sampling function $P_X : H_1 \to \mathbb{C}^m$ by

$$P_X(p) = \begin{bmatrix} p(x_1^{(1)}, \dots, x_1^{(n)}) \\ \vdots \\ p(x_m^{(1)}, \dots, x_m^{(n)}) \end{bmatrix}$$

and note that (2.1) shows the Bessel map is given by $F(w) = P_X(p_w)$. Because $F$ is invertible, $w$ is uniquely determined by $F(w)$. Hence any $p_w \in H_1$ is uniquely determined by its samples $P_X(p_w)$.

Conversely, if $X$ fails to be a frame for $V$, the mapping $F$ defined by (2.1) is still well-defined, but has non-trivial kernel $K$. In this case, $P_X(p_w) = P_X(p_{w+u})$ for all $u \in K$. So, in particular, $p_w$ is not uniquely determined from its samples at $x_1, \dots, x_m$.

A similar situation occurs for $k > 1$, where the space of interest is $\mathrm{Sym}^k(V)$. Since $\mathrm{Sym}^k(V)$ is spanned by pure tensor powers of elements in $V$ (see [11]) and a frame for a finite dimensional space is the same as a spanning set, it is valid to consider frames made up of pure tensor powers of elements of $V$. Let

$$X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$$

be a frame for $\mathrm{Sym}^k(V)$. As in the $k = 1$ case, mapping a polynomial to its coefficient sequence defines an isomorphism between $H_k(\mathbb{C}^n)$ and $\mathrm{Sym}^k(V)$ for $k > 1$. If $v = w^{\otimes k} \in \mathrm{Sym}^k(V)$ is a pure tensor power of $w \in V$, then, by virtue of (1.3), the Bessel map is

$$F^{(k)}(v) = \begin{bmatrix} \langle x_1^{\otimes k}, w^{\otimes k}\rangle \\ \vdots \\ \langle x_m^{\otimes k}, w^{\otimes k}\rangle \end{bmatrix} = \begin{bmatrix} \langle x_1, w\rangle^k \\ \vdots \\ \langle x_m, w\rangle^k \end{bmatrix} = \begin{bmatrix} p_v(x_1) \\ \vdots \\ p_v(x_m) \end{bmatrix},$$

where $p_v \in H_k$ is defined by $p_v(z) = \langle z, w\rangle^k$. Since $\mathrm{Sym}^k(V)$ is spanned by pure tensor powers of elements in $V$, for arbitrary $v \in \mathrm{Sym}^k(V)$, $F^{(k)}(v)$ is a vector of $m$ samples of a polynomial in $H_k$ taken at points $x_1, \ldots, x_m$. The coefficients of the polynomial are given by $v$ or $w$ which is uniquely determined by $F^{(k)}(v)$. Thus, as in the $k = 1$ case, polynomials in $H_k$ are uniquely determined by the samples $P_X^{(k)}(p) = [p(x_1), \ldots, p(x_m)]^\top$ if and only if $X^{(k)}$ is a frame for $\mathrm{Sym}^k(V)$. Together, this can be stated as the following theorem.

**Theorem 2.1.** *Let $k \geq 1$ and let $p$ be a homogeneous polynomial of degree $k$ in $n$ variables. It is possible to uniquely reconstruct $p$ from its samples at $m$ points $\{x_1, \ldots, x_m\}$ if and only if the set $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is a frame for the space* $\mathrm{Sym}^k(V)$.

At this point it is important to mention the reconstruction that is used for this purpose. Several examples are given in Section 3. Based on

$$X = \{x_1, \ldots, x_m\} \subset \mathbb{C}^n,$$

let

$$X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$$

be a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$, $\mathcal{F}^{(k)}$ be the corresponding frame operator, and

$$\{\widetilde{x_i^{\otimes k}} = \mathcal{F}^{(k)^{-1}} x_i^{\otimes k}\}_{i=1}^m$$

be the dual frame. If $p$ is a polynomial in $H_k(\mathbb{C}^n)$ whose samples at $X$ are known and whose coefficients are given by $c \in \mathrm{Sym}^k(\mathbb{C}^n)$, then, as given in (1.1), $c$ can be represented as

$$c = \sum_{i=1}^{m} \langle x_i^{\otimes k}, c \rangle \widetilde{x_i^{\otimes k}} = \sum_{i=1}^{m} p(x_i) \widetilde{x_i^{\otimes k}}. \tag{2.2}$$

In Theorem 2.3 below, it is shown that if one can reconstruct a polynomial in $H_k(\mathbb{C}^n)$ from a certain sampling set, then the same set can be used to reconstruct polynomials in $H_\ell(\mathbb{C}^n)$ for all $1 \leqslant \ell < k$. Conversely, almost every sampling set in $\mathbb{C}^n$ for $H_1$ gives rise to a sampling set for $H_k$ where $k > 1$ provided there are enough vectors in the set. The proof of Theorem 2.3 uses the following result given in [10].

**Theorem 2.2** ([10, Theorem 3.4]). *Let $\{x_1, \dots, x_m\}$ be a set of vectors in $\mathbb{C}^n$ that are in general position. Then the rank of the $k$-fold Hadamard product of the Grammian, $[\langle x_i, x_j \rangle^k]$, is*

$$\operatorname{rank}([\langle x_i, x_j \rangle^k]) = \binom{n+k-1}{k}$$

*provided $m \geq \binom{n+k-1}{k}$*

In the above, by being in general position, it is meant that no non-zero homogeneous polynomial of degree $k$ in $n$ variables can vanish at all of the vectors in $\{x_1, \dots, x_m\}$.

**Theorem 2.3.** *The following hold:*

(i) *If $X^{(k)} = \{x_1^{\otimes k}, \dots, x_m^{\otimes k}\}$ is a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$, then $X^{(\ell)}$ is a frame for $\mathrm{Sym}^\ell(\mathbb{C}^n)$ for all $1 \leqslant \ell < k$.*

(ii) *Almost every set of $m$ vectors in $\mathbb{C}^n$ such that $m \geqslant \binom{n+k-1}{k}$ results in a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$ for $k > 1$.*

*Proof.* (i) First consider $\ell = 1$. Suppose, to the contrary, that $X = \{x_1, \dots, x_m\}$ is not a frame for $\mathbb{C}^n$. Since $n$ is finite, this is possible if and only if there is a vector $w$ in $\mathbb{C}^n$ such that $w$ is orthogonal to each $x_i$, $i = 1, 2, \dots, m$. For such a $w$ one can construct $w^{\otimes k} \in \mathrm{Sym}^k(\mathbb{C}^n)$ and this satisfies

$$\sum_{i=1}^{m} |\langle x_i^{\otimes k}, w^{\otimes k} \rangle^2| = \sum_{i=1}^{m} |\langle x_i, w \rangle^{2k}| = 0.$$

This contradicts the fact that $X^{(k)}$ is a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. Thus $X$ must form a frame for $\mathbb{C}^n$ and the result holds for $\ell = 1$.

Since $X$ is a frame for $\mathbb{C}^n$, the vectors in $X$ are in general position. Otherwise, one would not be able to reconstruct $n$-variate homogeneous polynomials of degree $k$ from their samples at the elements in $X$ (see (2.2)) and Theorem 2.1 would fail. Also, $m \geq \binom{n+k-1}{k}$ since $X^{(k)}$ is a frame and hence a spanning set for $\mathrm{Sym}^k(\mathbb{C}^n)$. For $1 < \ell < k$, $\binom{n+\ell-1}{\ell} < \binom{n+k-1}{k}$. Thus $m > \binom{n+\ell-1}{\ell}$ and one can use Theorem 2.2 to say that

$$\mathrm{rank}([\langle x_i, x_j\rangle^\ell]) = \binom{n+\ell-1}{\ell}.$$

This means that the maximum number of linearly independent vectors in $X^{(\ell)}$ is $\binom{n+\ell-1}{\ell}$ which is the same as the dimension of $\mathrm{Sym}^\ell(\mathbb{C}^n)$, thereby making $X^{(\ell)}$ a frame for $\mathrm{Sym}^\ell(\mathbb{C}^n)$ when $1 < \ell < k$.

(ii) Almost every set of vectors $X = \{x_1, \dots, x_m\}$ in $\mathbb{C}^n$ is in general position (see [10]). By Theorem 2.2, the rank of the Grammian of $X^{(k)} = \{x_1^{\otimes k}, \dots, x_m^{\otimes k}\}$ is $\binom{n+k-1}{k}$ when $m \geqslant \binom{n+k-1}{k}$. This means that the maximum number of linearly independent vectors in the set $X^{(k)}$ is $\binom{n+k-1}{k}$ which is the same as the dimension of $\mathrm{Sym}^k(\mathbb{C}^n)$ and hence $X^{(k)}$ is a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. □

## 3 Illustrative examples

A vector in $\mathbb{C}^n$ can be extended to a vector in $\mathrm{Sym}^k(\mathbb{C}^n)$ by taking Kronecker products. Similar entries have to be identified so as to get the right dimension. For example, let $v = [a, b]^\mathsf{T} \in \mathbb{C}^2$. For $k = 2$, the Kronecker product is

$$[a, b]^{\otimes 2} = [a^2, ab, ba, b^2]^\mathsf{T}$$

and this is a tensor in $(\mathbb{C}^2)^{\otimes 2}$. To get the symmetric tensor with the same number of coordinates as the dimension of $\mathrm{Sym}^2(\mathbb{C}^2)$, $ab$ and $ba$ are identified to get $[a^2, ab, b^2]^\mathsf{T} \in \mathrm{Sym}^2(\mathbb{C}^2)$. This will be denoted by $v^{\odot 2}$. Thus starting from $v = [a, b]^\mathsf{T} \in \mathbb{C}^2$, the vector obtained in the three-dimensional space $\mathrm{Sym}^2(\mathbb{C}^2)$ is

$$v^{\odot 2} = [a^2, ab, b^2]^\mathsf{T}.$$

Similarly, the corresponding vector in the four-dimensional space $\mathrm{Sym}^3(\mathbb{C}^2)$ is

$$v^{\odot 3} = [a^3, a^2 b, ab^2, b^3].$$

In general, for a vector $v = [v^{(1)}, v^{(2)}, \dots, v^{(n)}] \in \mathbb{C}^n$, the corresponding vector $v^{\odot k} \in \mathrm{Sym}^k(\mathbb{C}^n)$ has $\binom{n+k-1}{k}$ coordinates.

As discussed in Section 2, if the samples of a homogeneous polynomial of degree $k$ are known at points that give rise to a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$, the polynomial can be uniquely reconstructed from these sample values by (2.2). This section discusses several examples illustrating this.

**Example 3.1.** Consider the space $V = \mathbb{C}^2$ over the field $\mathbb{C}$. Let $x_1 = [1, 0]^\mathsf{T}$, $x_2 = [0, 1]^\mathsf{T}$ and $x_3 = [1, 1]^\mathsf{T}$. The set $X = \{x_1, x_2, x_3\}$ is a frame for $V$ with corresponding Bessel map

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The frame operator is

$$\mathcal{F} = F^* F = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues of $\mathcal{F}$ are 1 and 3, which are the optimal lower and upper frame bounds respectively. Further,

$$\mathcal{F}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

which is the frame operator of the dual frame. The dual frame is denoted by

$$\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\},$$

where

$$\tilde{x}_1 = \mathcal{F}^{-1} x_1 = \left[\frac{2}{3}, -\frac{1}{3}\right]^\mathsf{T},$$

$$\tilde{x}_2 = \mathcal{F}^{-1} x_2 = \left[-\frac{1}{3}, \frac{2}{3}\right]^\mathsf{T},$$

$$\tilde{x}_3 = \mathcal{F}^{-1} x_3 = \left[\frac{1}{3}, \frac{1}{3}\right]^\mathsf{T}.$$

Consider reconstruction of the homogeneous polynomial $p$ of degree one in two variables defined by $p(u, v) = c^{(1)} u + c^{(2)} v$ from the three frame elements. Here $k = 1$, $n = 2$ and $m = 3$. Any $c = [c^{(1)}, c^{(2)}]^\mathsf{T} \in \mathbb{C}^2$ can be reconstructed via the reconstruction formula (2.2)

$$c = \sum_{i=1}^{3} \langle x_i, c \rangle \, \tilde{x}_i = \sum_{i=1}^{3} p(x_i) \tilde{x}_i. \tag{3.1}$$

If the samples of $p$, i.e., $p(x_1)$, $p(x_2)$ and $p(x_3)$ are known, then the coefficient vector $c = [c^{(1)}, c^{(2)}]^\mathsf{T}$ is found by calculating the right side of (3.1). This can be verified as follows. For this example,

$$p(x_1) = c^{(1)}, \quad p(x_2) = c^{(2)} \quad \text{and} \quad p(x_3) = c^{(1)} + c^{(2)}.$$

It can then be checked that the right side of (3.1), for the given frame, is

$$c^{(1)}\tilde{x}_1 + c^{(2)}\tilde{x}_2 + (c^{(1)} + c^{(2)})\tilde{x}_3 = [c^{(1)}, c^{(2)}]^\mathsf{T} = c.$$

This shows that the coefficients of $p(u, v)$ can be reconstructed from its samples at the frame elements.

**Example 3.2.** If the homogeneous polynomial to be reconstructed is of degree two as given by

$$p(u, v) = c^{(1)}u^2 + c^{(2)}uv + c^{(3)}v^2,$$

then one considers the space $\mathrm{Sym}^2(\mathbb{C}^2) \subset (\mathbb{C}^2)^{\otimes 2}$. The dimension of $\mathrm{Sym}^2(\mathbb{C}^2)$ is three, which is the same as the dimension of $H_2(\mathbb{C}^2)$. Hence at least three sampling points are needed. Consider the same set of sampling points as in Example 3.1, i.e.,

$$x_1 = [1, 0]^\mathsf{T}, \quad x_2 = [0, 1]^\mathsf{T} \quad \text{and} \quad x_3 = [1, 1]^\mathsf{T}.$$

One can extend this set to $(\mathbb{C}^2)^{\otimes 2}$ by taking Kronecker products. Restricting to $\mathrm{Sym}^2(\mathbb{C}^2)$ yields

$$x_1^{\odot 2} = [1, 0, 0]^\mathsf{T}, \quad x_2^{\odot 2} = [0, 0, 1]^\mathsf{T} \quad \text{and} \quad x_3^{\odot 2} = [1, 1, 1]^\mathsf{T}.$$

The Bessel map is

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

making the frame operator

$$\mathscr{F}^{(2)} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The minimum and maximum eigenvalues of $\mathscr{F}$ are 0.2679 and 3.7321, which are the optimal lower and upper frame bounds respectively. The frame operator for the dual frame is

$$\mathscr{F}^{(2)^{-1}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

making the dual frame

$$\widetilde{x_1^{\otimes 2}} = \mathscr{F}^{(2)^{-1}} x_1^{\otimes 2} = [1, -1, 0]^\mathsf{T},$$
$$\widetilde{x_2^{\otimes 2}} = \mathscr{F}^{(2)^{-1}} x_2^{\otimes 2} = [0, -1, 1]^\mathsf{T},$$
$$\widetilde{x_3^{\otimes 2}} = \mathscr{F}^{(2)^{-1}} x_3^{\otimes 2} = [0, 1, 0]^\mathsf{T}.$$

By the reconstruction formula (2.2) for $\mathrm{Sym}^2(\mathbb{C}^2)$, the coefficients of $p$ can be obtained from its samples at $x_1$, $x_2$, and $x_3$, i.e.,

$$[c^{(1)}, c^{(2)}, c^{(3)}]^\mathsf{T} = p(x_1)\widetilde{x_1^{\otimes 2}} + p(x_2)\widetilde{x_2^{\otimes 2}} + p(x_3)\widetilde{x_3^{\otimes 2}}. \tag{3.2}$$

To verify this one has to note that the polynomial

$$p(u,v) = c^{(1)}u^2 + c^{(2)}uv + c^{(3)}v^2$$

satisfies

$$p(x_1) = c^{(1)}, \quad p(x_2) = c^{(3)} \quad \text{and} \quad p(x_3) = c^{(1)} + c^{(2)} + c^{(3)}.$$

Inserting this in the right side of (3.2) gives $[c^{(1)}, c^{(2)}, c^{(3)}]^\mathsf{T}$.

**Example 3.3.** Consider now the frame for $\mathbb{C}^2$ formed by $x_1 = [1,0]^\mathsf{T}$, $x_2 = [2,0]^\mathsf{T}$ and $x_3 = [0,1]^\mathsf{T}$. In this case, reconstruction of

$$p(u,v) = c^{(1)}u^2 + c^{(2)}uv + c^{(3)}v^2$$

from samples $p(x_1)$, $p(x_2)$ and $p(x_3)$ is generally not possible, even though the number of samples is the same as the dimension of $H_2(\mathbb{C}^2)$. This is because $x_1$ and $x_2$ are scalar multiples of each other and the corresponding vectors in $\mathrm{Sym}^2(\mathbb{C}^2)$, the set $\{[1,0,0]^\mathsf{T}, [4,0,0]^\mathsf{T}, [0,0,1]^\mathsf{T}\}$ does not constitute a frame for $\mathrm{Sym}^2(\mathbb{C}^2)$. This is an example where the tensor powers of a frame for $V$ do not form a frame for $\mathrm{Sym}^k(V)$, even though the number of vectors is adequate.

**Example 3.4.** Reconstruction of homogeneous polynomials in $H_3(\mathbb{C}^2)$ requires at least four points, since the dimension of $\mathrm{Sym}^3(\mathbb{C}^2)$ and hence that of $H_3(\mathbb{C}^2)$ is four. Taking the frame

$$X = \{x_1, x_2, x_3, x_4\} = \{[1,0]^\mathsf{T}, [0,1]^\mathsf{T}, [1,1]^\mathsf{T}, [1,-1]^\mathsf{T}\}$$

for $\mathbb{C}^2$, computing Kronecker products, and restricting to $\mathrm{Sym}^3(\mathbb{C}^2)$ yields

$$\{x_1^{\odot 3}, x_2^{\odot 3}, x_3^{\odot 3}, x_4^{\odot 3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

A homogeneous polynomial of the form

$$p(u,v) = c^{(1)}u^3 + c^{(2)}u^2v + c^{(3)}uv^2 + c^{(4)}v^3$$

can be reconstructed from its samples at these points as

$$\begin{aligned} c^{(1)} &= p(1,0), \\ c^{(2)} &= \frac{1}{2}(p(1,1) + p(1,-1) - 2p(1,0)), \\ c^{(3)} &= \frac{1}{2}(p(1,1) + p(1,-1) - 2p(1,0)), \\ c^{(4)} &= p(0,1) \end{aligned}$$

so that $X^{(3)}$ constitutes a frame for $\mathrm{Sym}^3(\mathbb{C})^2$. The Bessel map is

$$F^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

making the frame operator

$$\mathcal{F}^{(3)} = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \end{bmatrix}.$$

The optimal lower and upper frame bounds are $A = 0.4384$ and $B = 4.5616$. The frame operator of the dual frame is

$$\mathcal{F}^{(3)^{-1}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3/2 & 0 & -1 \\ -1 & 0 & 3/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The dual frame is given by the vectors

$$\begin{aligned} \widetilde{x_1^{\otimes 3}} &= \mathcal{F}^{(3)^{-1}} x_1^{\otimes 3} = [1, 0, -1, 0]^{\mathsf{T}}, \\ \widetilde{x_2^{\otimes 3}} &= \mathcal{F}^{(3)^{-1}} x_2^{\otimes 3} = [0, -1, 0, 1]^{\mathsf{T}}, \\ \widetilde{x_3^{\otimes 3}} &= \mathcal{F}^{(3)^{-1}} x_3^{\otimes 3} = [0, 1/2, 1/2, 0]^{\mathsf{T}}, \\ \widetilde{x_4^{\otimes 3}} &= \mathcal{F}^{(3)^{-1}} x_4^{\otimes 3} = [0, -1/2, 1/2, 0]^{\mathsf{T}}. \end{aligned}$$

The coefficients of a degree-three homogeneous polynomial

$$p(u,v) = c^{(1)}u^3 + c^{(2)}u^2v + c^{(3)}uv^2 + c^{(4)}v^3$$

can be obtained by

$$[c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}]^{\mathsf{T}} = p(x_1)\widetilde{x_1^{\otimes 3}} + p(x_2)\widetilde{x_2^{\otimes 3}} + p(x_3)\widetilde{x_3^{\otimes 3}} + p(x_4)\widetilde{x_4^{\otimes 3}} \quad (3.3)$$

when one knows the sample values $p(x_i)$, $i = 1,2,3,4$, and the sampling set $X$ along with the degree of the polynomial.

**Example 3.5** (Mutually unbiased bases). Two orthonormal bases $B$ and $B'$ of $\mathbb{C}^n$ are said to be mutually unbiased when

$$|\langle b, b'\rangle|^2 = \frac{1}{n}$$

holds for every $b \in B$ and $b' \in B'$. It follows from results in [9] that the union of a maximal set of $n+1$ mutually unbiased bases (MUBs) of $\mathbb{C}^n$ with $n(n+1)$ elements gives rise to a tight frame for $\mathbb{C}^n$ and $\mathrm{Sym}^2(\mathbb{C}^n)$. For example, letting $\omega = e^{\frac{2\pi i}{3}}$, for $\mathbb{C}^3$, the following bases

$$\frac{1}{\sqrt{3}}\{(1,1,1), (1,\omega,\omega^2), (1,\omega^2,\omega)\},$$
$$\frac{1}{\sqrt{3}}\{(1,\omega,\omega), (1,\omega^2,1), (1,1,\omega^2)\},$$
$$\frac{1}{\sqrt{3}}\{(1,\omega^2,\omega^2), (1,\omega,1), (1,1,\omega)\}$$

together with the standard basis for $\mathbb{C}^3$ gives a maximal set of mutually unbiased bases in $\mathbb{C}^3$;

$$X = \frac{1}{\sqrt{3}}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix}, \begin{bmatrix}1\\\omega^2\\\omega\end{bmatrix}, \begin{bmatrix}1\\\omega\\\omega\end{bmatrix}, \begin{bmatrix}1\\\omega^2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\\omega^2\end{bmatrix}, \begin{bmatrix}1\\\omega^2\\\omega^2\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}1\\\omega\\1\end{bmatrix}, \begin{bmatrix}1\\1\\\omega\end{bmatrix}, \begin{bmatrix}\sqrt{3}\\0\\0\end{bmatrix}, \begin{bmatrix}0\\\sqrt{3}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\\sqrt{3}\end{bmatrix}\right\}. \quad (3.4)$$

These twelve vectors in (3.4) form a tight frame for $\mathbb{C}^3$ and can be made (by taking Kronecker products) into a tight frame for $\mathrm{Sym}^2(\mathbb{C}^3)$. However, these do not form a tight frame for $\mathrm{Sym}^3(\mathbb{C}^3)$. The Bessel map for $X$ is

$$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \\ 1 & \omega & \omega \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}.$$

The optimal lower and upper frame bounds for $X$ are both equal to 4. The frame for $\mathrm{Sym}^2(\mathbb{C}^3)$ that is generated from $X$ has both the optimal lower and upper bounds equal to 2. For the space $\mathrm{Sym}^3(\mathbb{C}^3)$ which is of dimension $\binom{3+3-1}{3} = 10$, one can get a corresponding set of twelve vectors from $X$ that are enough to form a spanning set but the smallest and largest non-zero eigenvalues of the frame operator are 1 and 2 respectively, implying that the frame tightness is lost.

**Example 3.6** (Kerdock sequences). Consider the space of $\ell$-dimensional binary vectors $\mathbb{Z}_2^\ell$, also called the *Hamming space*. Let $a$ and $b$ be any two vectors in $\mathbb{Z}_2^\ell$. For example, if $\ell = 2$, then $a, b \in \{[0,0]^\mathsf{T}, [0,1]^\mathsf{T}, [1,0]^\mathsf{T}, [1,1]^\mathsf{T}\}$. The so-called *first-order Reed–Muller functions*, also called the Walsh functions, are functions $\phi_{0,b} : \mathbb{Z}_2^\ell \to \mathbb{R}$ defined by

$$\phi_{0,b}(a) = \frac{1}{\sqrt{2^\ell}}(-1)^{b^T a}. \tag{3.5}$$

Fixing $b$ and ranging over all values of $a$ gives $2^\ell$ Walsh functions. These can be arranged as rows of a matrix (one row for each $b$) giving a Hadamard matrix $H_{2^\ell}$ of size $2^\ell$. For $\ell = 2$ one gets the $4 \times 4$ Hadamard matrix

$$H_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The Walsh functions form an orthonormal basis for $\mathbb{R}^{2^\ell}$ and $H_{2^\ell}$ are orthogonal matrices.

The *second order Reed–Muller functions* are parameterized by a binary symmetric matrix $P$ and defined as

$$\phi_{P,b}(a) = \frac{(-1)^{\mathrm{wt}(b)}}{\sqrt{2^\ell}} i^{(2b+Pa)^T a}. \tag{3.6}$$

It should be noted that the matrix $P$ is $\ell \times \ell$.

As with the Walsh functions where $P$ is zero, for a fixed binary symmetric matrix $P$, the set

$$\mathcal{F}_P = \{\phi_{P,b} : b \in \mathbb{Z}_2^\ell\}$$

forms an orthonormal basis for $\mathbb{C}^{2^\ell}$. For a fixed $\ell$, the total number of symmetric matrices is $2^{\ell(\ell+1)/2}$. These give a set of $2^{\ell(\ell+1)/2}$ orthonormal bases for $\mathbb{C}^{2^\ell}$. For each $P$ the vectors of the set $\mathcal{F}_P$ can be used as rows (or columns) to form a $2^\ell \times 2^\ell$ unitary matrix $U_P$, similar to the first-order case. By concatenating the matrices $U_{P_i}$, $i = 1, \ldots, \ell(\ell+1)/2$, one obtains the $2^\ell \times 2^{\ell(\ell+3)/2}$ matrix

$$\Phi_{RM} = [U_{P_1}, U_{P_2}, \ldots, U_{P_{2^{\ell(\ell+1)/2}}}] \tag{3.7}$$

whose columns are the Reed–Muller sequences. These columns form a tight frame for $\mathbb{C}^{2^\ell}$ with frame bounds $2^{\ell(\ell+1)/2}$. However, the tightness is not preserved when extended to $\mathrm{Sym}^k(\mathbb{C}^{2^\ell})$ by taking tensor powers of the columns.

It is known that the inner product between two different columns of the same matrix $U_{P_i}$ is zero. Fixing a vector $v_i$ which is the column of a certain matrix $U_{P_i}$ and letting another vector $v_j$ range over the columns of a different matrix $U_{P_j}$, the inner product

$$|\langle v_i, v_j\rangle| = \begin{cases} \frac{1}{\sqrt{2^r}}, & 2^r \text{ times}, \\ 0, & 2^\ell - 2^r \text{ times}, \end{cases} \tag{3.8}$$

where $r = \mathrm{rank}(P_i - P_j)$. When $r = \ell$, the vectors $v_i$ and $v_j$ come from matrices $P_i$ and $P_j$ respectively, such that the difference of these matrices has full rank. The set of symmetric matrices $P$ whose mutual differences have full rank is the *Kerdock set* [2,7,8]. For a given odd $\ell$, there are $\ell$ matrices of size $\ell \times \ell$ in the Kerdock basis making the size of the Kerdock set to be $2^\ell$. In this case, it is further implied by (3.8) that the bases given by such matrices $P$ result in mutually unbiased bases for $\mathbb{C}^{2^\ell}$. These $2^\ell$ bases combined with the standard basis for $\mathbb{C}^{2^\ell}$ form a set of mutually unbiased bases in $\mathbb{C}^{2^\ell}$, i.e., these $2^\ell(2^\ell + 1)$ vectors form a tight frame for $\mathbb{C}^{2^\ell}$ and can be made into a tight frame for $\mathrm{Sym}^2(\mathbb{C}^{2^\ell})$. For example, if $\ell = 3$, the Kerdock set gives a set of 72 vectors forming a tight frame for $\mathbb{C}^8$. These further give rise to a tight frame of 72 vectors for the $\binom{8+2-1}{2} = 36$-dimensional space $\mathrm{Sym}^2(\mathbb{C}^8)$. So this forms a good sampling set for homogeneous polynomials in eight variables of degrees one and two. Thereafter, these vectors cannot be used to sample higher order homogeneous polynomials since the dimension of $\mathrm{Sym}^3(\mathbb{C}^8)$ is $\binom{8+3-1}{3} = 120$ and 72 vectors are not enough to span $\mathrm{Sym}^3(\mathbb{C}^8)$.

**Remark 3.7.** As the degree $k$ or the dimension $n$ gets larger, numerical issues arise in calculating the inverse of the frame operator in order to get the dual frame that is needed for the reconstruction [5]. Since the upper and lower frame bounds determine the numerical merits of a particular frame, it is interesting to observe how starting with a fixed frame for $\mathbb{C}^2$ the frame bounds change with $k$ as the frame is extended to a frame for $\mathrm{Sym}^k(\mathbb{C}^2)$. A simple experiment with computing the eigenvalues of random symmetric matrices and their $k$-fold Hadamard product suggests that the ratio of the optimal upper and lower frame bounds or the ratio of the largest and smallest non-zero eigenvalues of these matrices can increase or decrease with successive products making the deduction inconclusive. Ideally, a sampling set is desired which guarantees frame tightness not just for $X = \{x_1, \dots, x_m\}$ but also for $X^{(\ell)} = \{x_1^{\otimes \ell}, \dots, x_m^{\otimes \ell}\}$, $1 < \ell \leqslant k$ for some $k$. The example of mutually unbiased bases as shown in Example 3.5 obviously satisfies this for $k = 2$, $n = 3$ and $m = 12$ since it results in a tight frame for $\mathrm{Sym}^2(\mathbb{C}^3)$. A similar result is true for the sequence obtained from the Kerdock set in Example 3.6. In general, sets which form a tight frame for $\mathrm{Sym}^\ell(\mathbb{C}^n)$ for all $1 \leqslant \ell \leqslant k$, often known as spherical $k$-designs, are hard to construct for $k > 2$.

## 4 Approximating functions by homogeneous polynomials

Let $\mathcal{P}_k(\mathbb{R}^n)$ denote the set of real $n$-variate algebraic polynomials of degree at most $k$. The theorem of Weierstraß asserts that for any compact $K \subset \mathbb{R}^n$, continuous $f$ in $C(K)$, and $m \in \mathbb{N}$, there is a polynomial $R_m \in \mathcal{P}_k(\mathbb{R}^n)$ such that $\lim_{m\to\infty} R_m = f$ uniformly in $K$. In [1] and [12], the authors have found similar results where $\mathcal{P}_k(\mathbb{R}^n)$ has been replaced by $H_k(\mathbb{R}^n)$, real $n$-variate homogeneous polynomials of degree $k$. The following theorem has been established in [12].

**Theorem 4.1** ([12]). *If $B \subset \mathbb{R}^n$ is the boundary of a convex domain and of class $\mathcal{C}^2_+$, which means that it is twice continuously differentiable and has strictly positive curvature, then for any even $f$ which is continuous and defined on $B$ and $\ell \in \mathbb{N}$, there is a homogeneous polynomial $p_{2\ell} \in H_{2\ell}(\mathbb{R}^n)$ such that*

$$\lim_{\ell\to\infty} p_{2\ell}(x) = f(x)$$

*uniformly on $B$.*

Results for odd and general continuous functions are also given in [12] but for convenience only even functions will be discussed here.

Having found a sampling criteria that guarantees unique reconstruction of a homogeneous polynomial of a known degree $k$, it is then desirable to use the likes of Theorem 4.1 to approximate a function, satisfying some smoothness condition, from appropriately chosen samples. This is the subject of the following discussion.

The goal is to find an approximation of an $n$-variate function $f$ from its samples at a set of points $X = \{x_1, \ldots, x_m\} \in \mathbb{R}^n$ i.e., from given $\{f(x_1), \ldots, f(x_m)\}$. The set $X$ is taken to form a frame for $\mathbb{R}^n$ and picked from the boundary $B$ of a convex domain on which $f$ is defined, as described in Theorem 4.1. Assuming that $f$ is even and satisfies other conditions of Theorem 4.1, it follows from Theorem 4.1 that given $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ and a homogeneous polynomial $p_{2n_1}$ of degree $2n_1$ such that

$$\|f - p_{2n_1}\|_\infty < \epsilon.$$

From (2.2), assuming that $m$ is big enough, the actual coefficients of $p_{2n_1}$ can be given by

$$c = \sum_{i=1}^{m} p_{2n_1}(x_i)\widetilde{x_i^{\otimes 2n_1}}.$$

As $p_{2n_1}$ is not known, one cannot find $\{p_{2n_1}(x_i)\}_{i=1}^m$ but suppose that $\{f(x_i)\}_{i=1}^m$ is given. Then it is possible to calculate

$$\tilde{c} = \sum_{i=1}^{m} f(x_i)\widetilde{x_i^{\otimes 2n_1}}.$$

Hence $\tilde{c}$ gives a polynomial $\tilde{p}_{2n_1}$ that is different from $p_{2n_1}$. If $\tilde{p}_{2n_1}$ is to be used as an approximation of $f$ on $B$, then the error can be estimated by bounding

$$\begin{aligned}\|\tilde{p}_{2n_1}(x) - f(x)\| &= \|\tilde{p}_{2n_1}(x) - p_{2n_1} + p_{2n_1} - f(x)\| \\ &\leqslant \|\tilde{p}_{2n_1}(x) - p_{2n_1}\| + \|p_{2n_1} - f(x)\|,\end{aligned}$$

where $\|\cdot\|$ is some suitably chosen norm. Due to Theorem 4.1, the second term can be easily bounded by a small quantity; the problem is to bound the first term.

**Example 4.2.** Consider $\mathbb{R}^2$ and let $f(x, y)$ be a continuous even function defined on $B = \{(x, y) : x^2 + y^2 = 1\}$, the unit circle. Considering the $L^2$-norm, from Theorem 4.1, it follows that given $\epsilon_1$, there exists some $n_1 \in \mathbb{N}$ such that

$$\|f - p_{2n_1}\|_{L^2(B)} < \epsilon_1$$

for some homogeneous polynomial $p_{2n_1}$ of degree $2n_1$. The dimension of the space of bivariate homogeneous polynomials of degree $2n_1$ is

$$\binom{2+2n_1-1}{2n_1} = 2n_1 + 1.$$

Let $X = \{x_1, \ldots, x_m\}$ be a set of points on the unit circle that forms a frame for $\mathbb{R}^2$ where $m \geq 2n_1 + 1$. The actual coefficients of $p_{2n_1}$ are given by

$$c = \sum_{i=1}^{m} p_{2n_1}(x_i)\widetilde{x_i^{\otimes 2n_1}}$$

where $c = [c^{(1)}, \ldots, c^{(2n_1+1)}]^\mathsf{T}$. Since only $f(x_i)$ are available, one is able to calculate

$$\tilde{c} = \sum_{i=1}^{m} f(x_i)\widetilde{x_i^{\otimes 2n_1}}$$

where $\tilde{c} = [\tilde{c}^{(1)}, \ldots, \tilde{c}^{(2n_1+1)}]^\mathsf{T}$. This gives us another polynomial $\widetilde{p}_{2n_1}$. Note that the polynomials $p_{2n_1}$ and $\widetilde{p}_{2n_1}$ have degree $2n_1 + 1$ and can be written as

$$p_{2n_1} = \sum_{j=0}^{2n_1} c^{(j)} x^{2n_1-j} y^j, \quad \widetilde{p}_{2n_1} = \sum_{j=0}^{2n_1} \tilde{c}^{(j)} x^{2n_1-j} y^j.$$

Suppose that one would like to use $\widetilde{p}_{2n_1}$ as an approximation of $f$. In the $L^2$-norm we get

$$\|\widetilde{p}_{2n_1} - f\|_{L^2(B)} \leqslant \|\widetilde{p}_{2n_1} - p_{2n_1}\|_{L^2(B)} + \|p_{2n_1} - f\|_{L^2(B)} \tag{4.1}$$

and the second term on the right of (4.1) is bounded by $\epsilon_1$. The square of the first term on the right of (4.1) is

$$\begin{aligned}
\|p_{2n_1} - \widetilde{p}_{2n_1}\|^2_{L^2(B)} &= \int_B |p_{2n_1} - \widetilde{p}_{2n_1}|^2 ds \\
&= \int_B \left| \sum_{j=0}^{2n_1} (c^{(j)} - \tilde{c}^{(j)}) x^{2n_1-j} y^j \right|^2 ds \\
&= \int_0^{2\pi} \left| \sum_{j=0}^{2n_1} (c^{(j)} - \tilde{c}^{(j)}) \cos^{2n_1-j}(t) \sin^j(t) \right|^2 dt \\
&\leq \int_0^{2\pi} \left( \sum_{j=0}^{2n_1} |c^{(j)} - \tilde{c}^{(j)}| \right)^2 dt = \left( \sum_{j=0}^{2n_1} |c^{(j)} - \tilde{c}^{(j)}| \right)^2 2\pi.
\end{aligned}$$

By Cauchy–Schwarz

$$\left(\sum_{j=0}^{2n_1}|c^{(j)}-\tilde{c}^{(j)}|\right)^2 \le (2n_1+1)\sum_{j=0}^{2n_1}|c^{(j)}-\tilde{c}^{(j)}|^2.$$

Further,

$$\begin{aligned}\sum_{j=0}^{2n_1}|c^{(j)}-\tilde{c}^{(j)}|^2 &= \|c-\tilde{c}\|_{\ell_2}^2\\ &= \left\|\sum_{i=1}^{m} p_{2n_1}(x_i)\widetilde{x_i^{\otimes 2n_1}} - \sum_{i=1}^{m} f(x_i)\widetilde{x_i^{\otimes 2n_1}}\right\|_{\ell_2}^2\\ &= \left\|\sum_{i=1}^{m}[p_{2n_1}(x_i)-f(x_i)]\widetilde{x_i^{\otimes 2n_1}}\right\|_{\ell_2}^2\\ &\le \epsilon^2\left\|\sum_{i=1}^{m}\widetilde{x_i^{\otimes 2n_1}}\right\|_{\ell_2}^2\\ &\le \frac{\epsilon^2}{A_{n_1}^2}\left(\sum_{i=1}^{m}\|x_i^{\otimes 2n_1}\|_{\ell_2}\right)^2\\ &= \frac{\epsilon^2 m^2}{A_{n_1}^2}\end{aligned}$$

where $A_{n_1}$ is the lower frame bound for $\{x_1^{\otimes 2n_1},\dots,x_m^{\otimes 2n_1}\}$, $\epsilon$ is a small quantity depending on $\epsilon_1$ due to the approximation of $f$ by $p_{2n_1}$, and one uses the fact that $\|x_i^{\otimes 2n_1}\|_{\ell_2} = 1$ for $i = 1,\dots,m$. Taking $m = 2n_1+1$,

$$\|p_{2n_1}-\widetilde{p}_{2n_1}\|_{L^2(B)} \le \sqrt{2\pi}\,\frac{\epsilon}{A_{n_1}}(2n_1+1)^{3/2}.$$

Putting this all together one gets the error estimate of the approximation as

$$\|\widetilde{p}_{2n_1}(x)-f(x)\|_{L^2(B)} \leqslant \sqrt{2\pi}\,\frac{\epsilon}{A_{n_1}}(2n_1+1)^{3/2}+\epsilon_1.$$

The frame bound $A_{n_1}$ can increase or decrease with $n_1$ and for a small approximation error one would want $\frac{(2n_1+1)^{3/2}}{A_{n_1}}$ to tend to zero as $n_1$ goes to infinity.

The above example shows that for a satisfactory approximation, of a given function by homogeneous polynomials, there are some requirements that the underlying frame needs to satisfy. Constructing such frames has not been discussed here and will constitute future work.

## 5 Conclusions

It has been shown that homogeneous polynomials of known degree $k$ in $n$ variables can be uniquely reconstructed from their samples at elements that give rise to a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. Such a set can also be used to reconstruct $n$-variate homogeneous polynomials of all degrees $\ell$ where $1 \leqslant \ell < k$. In recent work [1, 12] conditions under which a smooth function can be approximated by homogeneous polynomials have been established. Combining these results, a scheme to approximately reconstruct smooth functions from sampled data has been proposed.

**Acknowledgments.** The authors would like to thank John McDonald for useful discussions on the topic of this paper. The first named author is also indebted to Hirotachi Abo for useful discussions on symmetric tensors.

## Bibliography

[1] D. Benko and A. Kroó, A Weierstrass-type theorem for homogeneous polynomials, *Transactions of the American Mathematical Society* **361** (2009), 1645–1665.

[2] R. Calderbank, personal communication, 2009.

[3] O. Christensen, *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhäuser-Verlag, Boston, 2003.

[4] S. Datta, S. D. Howard and D. Cochran, Geometry of the Welch bounds, *Linear Algebra and its Applications* **437** (2012), 2455–2470.

[5] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics 61, SIAM, Philadelphia, 1992.

[6] C. T. J. Dodson and T. Poston, *Tensor Geometry: The Geometric Viewpoint and Its Uses*, Second edition, Graduate Texts in Mathematics 130, Springer-Verlag, Berlin, 1991.

[7] A. R. Hammons Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Transactions on Information Theory* **40** (1994), 301–319.

[8] A. M. Kerdock, A class of low rate nonlinear binary codes, *Information and Control* **20** (1972), 182–187.

[9] A. Klappenecker and M. Rötteler, Mutually unbiased bases are complex projective 2-designs, in: *ISIT 2005. Proceedings of the International Symposium on Information Theory* (Adelaide 2005), IEEE Press, Piscataway (2005), 1740–1744.

[10] I. Peng and S. Waldron, Signed frames and Hadamard products of Gram matrices, *Linear Algebra and its Applications* **347** (2002), 131–157.

[11] R. Shaw, *Linear Algebra and Group Representations. Volume 2: Multilinear Algebra and Group Representations*, Academic Press, London, 1983.

[12] P. Varjú, Approximation by homogeneous polynomials, *Constructive Approximation* **26** (2007), 317–337.

[13] A. Zakhor and G. Alvstad, Two-dimensional polynomial interpolation from nonuniform samples, *IEEE Transactions on Signal Processing* **40** (1992), 169–180.

Received September 22, 2012; accepted January 4, 2013.

**Author information**

Somantika Datta, Department of Mathematics, University of Idaho,
Moscow ID 83844-1103, USA.
E-mail: sdatta@uidaho.edu

Stephen Howard, Defense Science and Technology Organisation,
PO Box 1500, Edinburgh 5111 SA, Australia.
E-mail: Stephen.Howard@dsto.gov.au

Douglas Cochran, School of Mathematical and Statistical Sciences,
Arizona State University, Tempe AZ 85287-1804, USA.
E-mail: cochran@asu.edu